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EXTENSIONS OF MULTIPLICATIVE SYSTEMS AND MODULAR LATTICES

BY

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In my paper on Extensions of Partially ordered sets-II,* the question of the existence of a canonical extension to various combinations of operations was considered, when the given basic ordered set was quite arbitrary. Now, in this paper, we take the basic set to be one of the particular types: a multiplicative system, an additive system, a lattice, or a modular lattice (with units, in all cases). For such ordered sets the existence question is definitively settled for all combinations of operations chosen from the following set, except for a few combinations which require further analysis: $[0, 1, +, \Sigma, \bar{+}, \bar{\Sigma}, \times, \pi, \bar{\times}, \bar{\pi}, P, S]$. For these exceptional combinations also the existence question is resolvable when the ordered set given is finite.

The final analysis of the various combinations in § 5 is based on the constructed extensions of § 1 and the examples given in §§ 2-4 proving non-existence of a canonical or minimal or even any extension.

1. Extension by $\bar{\Sigma}$-ideals. This first extension of a multiplicative system, K, with units is essentially the same as the one given by MacNeille.† The elements of the extension, L, consist of the $\bar{\Sigma}$-ideals of K, and set-inclusion partially orders these elements.‡

* This *Journal*, Vol. XII, (1948); this is referred to hereafter as 'EII'.'EI' is the first part, which appeared in Vol. XI, (1947).

† 'Partially Ordered Sets', *Trans. Amer. Math. Soc*, Vol. 42. This is referred to hereafter as '*M*'.

‡ A non-null subset A of K is a $\bar{\Sigma}$-ideal if (i) $a < b$ and $b \varepsilon A$ imply $a \varepsilon A$; and (ii) $a = \bar{\Sigma} a_i$ in K, $a_i \varepsilon A$ imply $a \varepsilon A$.

The (1, 1) correspondence $a \rightleftarrows P_\mu(a)$ gives an isomorphism between K and the subset, K' of L consisting of the principal (μ-) ideals of K. It is easily verified that L is an extension of K to the set of operations $\psi = \{0, 1, \mp, \bar{\Sigma}, \bar{\times}, \times, \pi, P\}$ with K' as image of K [that is, L is closed for the operations in ψ and is further a supersystem of K' for these operations; proofs may be found in 'M']. In L the sum and product of elements (A_ι) are given by:

ΣA_ι = set of elements of K expressible as distributive sums (in K) of elements chosen from the set-union ($\cup A_\iota$).

πA_ι = the set-intersection ($\cap A_\iota$).

We now prove even more; namely,

THEOREM 1. *The extension, L, of K consisting of the $\bar{\Sigma}$-ideals of K is a canonical extension of K to ϕ, if ϕ satisfies the conditions*:

$\bar{\Sigma}$ and ($\times$ or π) are contained in ϕ, and $\phi \subset \psi = \{0, 1, \mp, \bar{\Sigma}, \bar{\times}, \times, \pi, P\}$.

PROOF. As L is an extension of K to ψ, so is it for any subset ϕ of ψ. By the theorem of 4.342 ('E I'), it is enough to prove that L is a canonical extension to $\phi = (\times, \bar{\Sigma})$. For this, we assign representations to the elements, A, of L in terms of ϕ and K' as follows:

$$A = \bar{\Sigma} P_\mu(a_\iota), \; a_\iota \varepsilon A.$$

If $B = \bar{\Sigma} P_\mu(b_j) - b_j \varepsilon B$ is another element of L, then the order relation between A, B is required by these representations in all extensions of K to ϕ. For suppose $\bar{L}$ is another extension of K to ϕ with $\bar{K}$ as image of K (the isomorphism between K, $\bar{K}$ taking a of K into $\bar{a}$ of $\bar{K}$). If

$$\alpha = \bar{\Sigma}\, \bar{a}_\iota, \; a_\iota \varepsilon A, \;\; \beta = \bar{\Sigma}\, \bar{b}_j, \; b_j \varepsilon B$$

are the elements of $\bar{L}$ with representations similar to A, B in L then $A \subset B$ implies that every a_ι is a b_j, and so every $\bar{a}_\iota$ is a $\bar{b}_j$; whence, $\Sigma \bar{a}_\iota < \Sigma \bar{b}_j$, or $\alpha < \beta$.

While, conversely, if $\alpha < \beta$, then, for any $\bar{a}_\iota$, $\bar{a}_\iota < \bar{\Sigma} \bar{a}_\iota = \alpha < \beta = \bar{\Sigma} b$, so that $\bar{a}_\iota = \bar{a}_\iota (\bar{\Sigma} \bar{b}_j) = \bar{\Sigma}(\bar{a}_\iota \bar{b}_j)$ in $\bar{L}$. As $\bar{L}$ is an

extension of K to ϕ, and ϕ contains $\times$ and $\bar{\Sigma}$, $\bar{L}$ is a supersystem of $\bar{K}$ to $\times$ and $\bar{K}$ (like K) is closed for $\times$. Hence $\bar{a}_i.\bar{b}_j$ is the same in $\bar{K}$ and $\bar{L}$, and $\bar{a}_i = \Sigma(\bar{a}_i.\bar{b}_j)$ in $\bar{L}$ implies $\bar{a}_i = \Sigma(\bar{a}_i.\bar{b}_j)$ in $\bar{K}$ also. By similar reasoning, if $\bar{d} \varepsilon \bar{K}$, $\bar{d}.\bar{a}_i = \bar{d}.\bar{\Sigma}(\bar{a}_i.\bar{b}_j) = \bar{\Sigma}(\bar{d}.\bar{a}_i.\bar{b}_j)$ in $\bar{L}$ implies $\bar{d}.\bar{a}_i = \Sigma(\bar{d}.\bar{a}_i.\bar{b}_j)$ in $\bar{K}$. This proves that in $\bar{K}$, $\bar{a}_i = \bar{\Sigma}(\bar{a}_i.\bar{b}_j)$; by the isomorphism between K and $\bar{K}$, it follows that $a_i = \bar{\Sigma}(a_i.b_j)$ in K. Since $a_i.b_j < b_j$, each $(a_i.b_j) \varepsilon B$ and so $\bar{\Sigma}(a_i.b_j) \varepsilon B$. Thus each a_i of A is in B, or $A \subset B$. We have now proved that the relation between A, B is required in all extensions of K to ϕ.

Finally, to prove the separation condition: if $A \neq B$, we may assume $A \not\subset B$. There is then an element a of K in A but not in B. That is, $P_\mu(a)$ of K' is $\subset A$, but not $\subset B$. This element $P_\mu(a)$ of K' then separates the distinct elements A, B of L. Hence our theore i is proved completely.

We can use the 'resolution' theorem of 'E I' and deduce

THEOREM 2. *The subset, M, of the extension L, of Theorem* 1, *consisting of the $\bar{\Sigma}$-ideals which are obtainable as finite sums in L, of elements of K' forms a canonical extension of K to θ, with K' as image of K, if $(\bar{+}, \times) \subset \theta \subset (0, 1, \bar{+}, \bar{\times}, \times)$.*

PROOF. By the resolution theorem, it is only necessary to show that the relations between elements A, B of M will be required in extensions of K to θ by their representations as finite distributive sums. Now let $A = \bar{\Sigma}\, P_\mu(a_i)$, $i = 1, \ldots, n$, and $B = \bar{\Sigma} P_\mu(b_j)$, $j = 1, \ldots, m$, in L, (and M), while $\alpha = \bar{\Sigma}\bar{a}_i$, $\beta = \bar{\Sigma}\,\bar{b}_j$ are elements with similar representations in an extension $\bar{L}$ of K to θ with $\bar{K}$ as image of K. Then, as in the proof of the Theorem 1, we can see that $\alpha \leqslant \beta$ implies $A \subset B$. Conversely if $A \subset B$, then $a_i \varepsilon A \subset B$ implies $a_i = \bar{\Sigma}\, c_\mu$ in K, where each c_μ is < some b_j and also $< a_i$. Hence, as K is closed for $\times$, it follows that $a_i = \bar{\Sigma}(a_i.b_j)$ in K. Therefore, in the isomor-

phic ordered set $\bar{K}$ also, $\bar{a}_i = \sum_j(\bar{a}_i.\bar{b}_j)$. Since $\bar{L}$ is a super-system of $\bar{K}$ for $\times$ and $\bar{+}$, $\bar{a}_i = \sum_j(\bar{a}_i.\bar{b}_j)$ in $\bar{L}$ also. Hence $\bar{a}_i < \sum_j \bar{b}_j = \beta$. As this is true for each $\bar{a}_i$, so $\sum \bar{a}_i < \beta$ or $\alpha < \beta$.

2. CASES OF NO MINIMAL EXTENSION. For proving that no minimal extension of a given ordered set K need exist to a set of operations ϕ, we proceed as in 'E II' to construct two extensions $\bar{L}$, L' such that no extension of K to ϕ can be at the same time $< \bar{L}$ and $< L'$. The ordered sets $K, \bar{K}, L'$ are described by assigning their elements and sufficient relations of order between them to deduce their structures. The images $\bar{K}$, K' of K in $\bar{L}$, L' are indicated by naming the elements corresponding to an element a in K by $\bar{a}$, a' respectively.

2.1. *A multiplicative system K need have no minimal extension to ϕ if $(+) \subset \phi \subset (0, 1, +, \times, \bar{P}, S)$.*

K: $0 < a <$ each of the elements $a_1, a_2, a_3, \ldots < \ldots$ $\ldots b_n < b_{n-1} < \ldots < b_1 < 1$;

$\bar{L}$: $\bar{0} < \bar{a} <$ each of the elements $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots < \bar{b} < \ldots$ $\ldots \bar{b}_n < \bar{b}_{n-1} < \ldots < \bar{b}_1 < \bar{1}$;

L: $0' < a' <$ each of the elements $a_1', a_2', \ldots$; L' contains all finite subsets, F', of the set $(a_1', a_2', \ldots)$ and these are partially ordered by set-inclusion; the one-element sets $(a_1'), (a_2') \ldots$ are replaced by the single elements $a_1', a_2', \ldots$; and each $F' < \ldots b_n' < b_{n-1}' \ldots$ $< b_1' < 1'$.

The reader may verify that $\bar{L}$, L' are extensions of K to $(0, 1, +, \times, P, S)$ with $\bar{K}$, K' as images of K. If now an extension of K to ϕ, containing $+$, were to be $< \bar{L}$, it must be *super-isomorphic** to $\bar{L}$, since any subset of $\bar{L}$ containing $\bar{K}$ and closed for $+$ must be necessarily $\bar{L}$ itself, ($\bar{b}$ being necessary as the sum of any two of the $\bar{a}_i$). But

* A *super-isomorphism* betweens two extension $\bar{L}$, L' is an isomorphism which carries the images of K in $\bar{L}$, L' one into the other; (see E I).

$\bar{L}$ cannot be $< L'$ (as L' has no element, to correspond to $\bar{b}$, which is $<$ every b_i' and $>$ each a_i'); and so an extension L to ϕ cannot be $< L'$ if it is $< \bar{L}$.

2.2. *A multiplicative system, K, need not have a minimal extension to ϕ, if (i) $\phi \subset [0, 1, +, \Sigma, \bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}, P, S]$ and (ii) ϕ contains one of $\mp$, $\bar{\times}$, $\bar{\Sigma}$, $\bar{\pi}$ or P.*

This is the dual of the example given in §8.1, E II. The K, $\bar{L}$, L' for this can be obtained from those of §2.3 below, by replacing the 1 in each case by an infinite descending chain, without a least element for K, but with least elements for $\bar{L}$ and L'.

2.3. *A modular lattice with units need not have a minimal extension to ϕ, if (i) $\phi \subset (0, 1, \mp, \bar{\Sigma}\,\bar{\times}, \bar{\pi}, P, S)$; and (ii) ϕ contains one of $\mp$, $\bar{\Sigma}$, $\bar{\times}$, $\bar{\pi}$, P or S.*

K: $0 <$ each of $a, b, c < 1$.

$\bar{L}$: $\bar{0} <$ each of $\bar{a}, \bar{b}, \bar{c}$; $\bar{b}$ and $\bar{c} < \bar{p}$, $\bar{c}$ and $\bar{a} < \bar{q}$, $\bar{a}$ and $\bar{b} < \bar{r}$; each of $\bar{p}, \bar{q}, \bar{r} < \bar{1}$.

L': $0' <$ each of d' e', f'; e' and $f' < a'$, f' and $d' < b'$, d' and $e' < c'$; each of $a', b', c' < 1'$.

[See the lattice diagrams in E. I, p. 6, figs. 5, 6, 7].

It can be verified that $\bar{L}$, L' are extensions of K to $(0, 1, \mp, \bar{\Sigma}\ \bar{\times}, \bar{\pi}, P, S)$ and any extension of K to a subset of these containing $\bar{+}, \bar{\Sigma}, \bar{\times}, \bar{\pi}$, P or S must be superisomorphic to $\bar{L}$ if it is $< \bar{L}$. and as $\bar{L}$ is not $< L'$, such an extension cannot be $< L'$ if it is $< \bar{L}$.

2.4. *A modular lattice with units need not have a minimal extension to ϕ, if $P \varepsilon \phi \subset (1, \times, \pi, P)$ or $S \varepsilon \phi \subset (0, +, \Sigma, S)$.*

This is proved by the example of §6.4, E II, and its dual.

3. A CASE OF NO CANONICAL EXTENSION.

A modular lattice K with units need not have a canonical extension to ϕ when $\phi \subset (0, 1, \times, \pi, \bar{\times}, \bar{\pi}, P)$ and contains $\bar{\times}$ or $\bar{\pi}$; or dually.

K: $0 < l <$ each of $a, b, c < 1$.

$\bar{L}$: $\bar{0} < \bar{l} < \bar{a} <$ each of $\bar{q}, \bar{r} < \bar{1}$;

$\bar{o} < \bar{l} < \bar{b} <$ each of $\bar{r}, \bar{p} < \bar{1}$;
$\bar{o} < \bar{l} < \bar{c} <$ each of $\bar{p}, \bar{q} < \bar{1}$.

$\bar{L}$ is an extension of K to (o, 1, $\times$, π, $\overline{\times}$, $\bar{\pi}$, P) and also the only subset of $\bar{L}$ containing $\bar{K}$ and closed for $\overline{\times}$ or $\bar{\pi}$. So if K were to have a canonical extension to ϕ containing $\overline{\times}$ or $\bar{\pi}$, the canonical extension must be isomorphic to $\bar{L}$. But in $\bar{L}$ the elements $\bar{p}$, $\bar{q}$, $\bar{r}$ cannot be assigned any representation as required elements in terms of K and ϕ, since the only elements of $\bar{L}$ that are obtained by using the operations from ϕ on subsets of $\bar{K}$ are in $\bar{K}$ itself. For a dual set of operations a similar proof can be given, or we can appeal to the duality theorem (§4.341, E I).

4. Cases of modular lattices with no extension to certain sets of operations.

4.1. *A modular lattice with units may not have any extension at all to ϕ, if ϕ contains one operation at least from each of the three sets (i) $+$, Σ, (ii) $\times$, π, (iii) $\overline{+}$, $\bar{\Sigma}$, $\overline{\times}$, $\bar{\pi}$.*

It is enough to choose a modular, non-distributive lattice, like K in the last section. In K there are then elements a, b, c with $a(b+c) >, \neq, a.b+a.c$. In any extension L of K to a ϕ as above, if we replace the image of K by K itself, then in L also $a.b+a.c$, $a.(b+c)$ are distinct elements; hence L contains a non-distributive sum, which is not possible if K were closed for ϕ (as closure for ϕ implies that L is a distributive lattice).

4.2. *A modular lattice K with units may have no extension at all to ϕ if ϕ contains either (i) P and ($+$ or Σ) or (ii) S and ($\times$ or π).*

K: $o < d <$ each of $b, c, e < f < 1$;
$o < a < e < g < 1$.

[This is as illustrated in §7.1, E II, p. 104, with the addition of a line joining the points representing d and e].

If K had an extension L to P and ($+$ or Σ) then, (taking K itself as the image of K in L), $a = P(b) = P(c)$ in L, as $a = P(b) = P(c)$ in K; and so $a.b = a.c = o$ and

b, c are $< P(a)$ in L. Then $P(a)$ must be $> (b+c)$ in L; but $f = b+c$ in L, if L is an extension, and so a super-system, of K to ($+$ or Σ). Then it is impossible that $P(a) > f$, as $f > a$ also, and $P(a) > a$ would imply $a = 0$ contrary to the structure of K. Thus K has no extension to P and ($+$ or Σ). The dual result is proved similarly.

4.3. *A distributive lattice, K, with units may have no extension at all to ϕ if ϕ contains either (i) π, $\bar{\pi}$ and ($\mp$ or $\bar{\Sigma}$), or (ii) Σ, $\bar{\Sigma}$ and ($\overline{\times}$ or $\bar{\pi}$).*

It is enough to choose a distributive lattice K with an infinite product ($\pi\, b_\iota$) not distributed by an element a in it; so that $a + \pi b_\iota < \pi(a+b_\iota)$, but $a + \pi b_\iota \not> \pi(a+b_\iota)$. Then $a \mp \pi b_\iota = a + \pi b_\iota <, \not>, \pi(a+b_\iota) = \bar{\pi}(a \mp b_\iota)$. In any extension, L, of this K for π and ($\mp$ or $\bar{\Sigma}$), $a \mp \pi b_\iota <, \not>, \pi(a \mp b_\iota)$, (if we replace the image of K by K itself). So πb_ι is not distributive in L, and L cannot be closed for, nor an extension of K for, $\bar{\pi}$ also.

Dually we can deduce that a distributive lattice with a non-distributive infinite sum cannot have any extension for a set of operations including Σ, $\bar{\Sigma}$ and ($\overline{\times}$ or $\bar{\pi}$).

5. SUMMARY OF RESULTS.

5.1. For convenience we shall first group the subsets, ϕ, of ($0, 1, +, \Sigma, \times, \pi, \mp, \bar{\Sigma}, \overline{\times}, \bar{\pi}, P, S$) into various types or classes as follows: [the dual being marked with a dash].

(1) $\phi \subset (0, 1, \times)$;

(2) $(\Sigma \text{ or } \pi) \,\varepsilon\, \phi \subset (0, 1, +, \Sigma, \times, \pi)$;

(3) $(0, P, \pi) \subset \phi \subset (0, 1, \times, \pi, P)$;

(4) $\bar{\Sigma}$ and $(\times \text{ or } \pi) \,\varepsilon\, \phi \subset (0, 1, \times, \pi, \overline{\times}, \mp, \bar{\Sigma}, P)$;

(5) $(\times, \mp) \subset \phi \subset (0, 1, \times, \mp, \overline{\times})$;

(6) $\mp$ and $(P \text{ or } \pi) \,\varepsilon\, \phi \subset (0, 1, \times, \pi, \times, \mp, P)$;

(7) $\times$, $\bar{\pi}$, and $(\mp, \text{ or } \bar{\Sigma}) \,\varepsilon\, \phi \subset (0, 1, \times, \mp, \bar{\Sigma}, \overline{\times}, \bar{\pi}, P)$.

In terms of these groups we can now state our conclusions from the constructions and examples of the last four sections, as follows:

THEOREM 3. (*i*) *For a multiplicative system with units there exist canonical extensions to* ϕ, *if* ϕ *belongs to any of the types* (1) *to* (5) *or* (3′).*

(*ii*) *A multiplicative system with units need not have a canonical extension to* ϕ, *if* ϕ *does not belong to any one of the types* (1) *to* (7) *or* (3′).

THEOREM 4. (*i*) *For a lattice, or modular lattice, with units there exist canonical extensions to* ϕ, *if* ϕ *belongs to any one of the types* (1) *to* (5) *or their duals* (1′) *to* (5′).

(*ii*) *A lattice, or modular lattice, with units need not have a canonical extension to* ϕ *if* ϕ *does not belong to any of the types* (1) *to* (7) *or any of their duals* (1′) *to* (7′).

PROOF OF THEOREM 3. (*i*) *The existence cases:*

(1) If $\phi \subset (0, 1, \times)$, a multiplicative system with units is closed for ϕ and so it would be its own canonical extension to ϕ.

(2) If Σ or $\pi \varepsilon \phi \subset (0, 1, \times, \pi, +, \Sigma)$, then we have already proved the existence of a canonical extension to ϕ of any partially ordered set with units (see E II, §5.2); this covers, as a special case, multiplicative systems with units.

(3) If $(0, P, \pi) \varepsilon \phi \subset (0, 1, \times, \pi, P)$, then we have proved before that there is a canonical extension of any ordered set with units to ϕ (see E II, §6.2). That result includes the multiplicative system with units also. Dually, there is a canonical extension of any ordered set with units to ϕ, if $(1, S, \Sigma) \varepsilon \phi \subset (0, 1, +, \Sigma, S)$.

(4) If $\bar{\Sigma}$ and ($\times$ or π) $\varepsilon \phi \subset (0, 1, \times, \pi, \overline{\times}, \overline{+}, \bar{\Sigma}, P)$, then Theorem 1 of § 1 above proves the existence of a canonical extension to ϕ of any multiplicative system with units.

(5) If $(\times, \overline{+}) \subset \phi \subset (0, 1, \times, \overline{+}, \overline{\times})$, then, in Theorem 2 of § 1 above, we have proved that any multiplicative system with units does have a canonical extension to ϕ.

* (3′) is $(1, S, \Sigma) \subset \phi \subset (0, 1, +, \Sigma, S)$.

(*ii*) *The non-existence cases.*

We divide the cases of ϕ to be considered into the following categories:

(*a*) ϕ does not contain ($\times$ or π) but contains ($+$ or Σ);

(*b*) ϕ contains ($+$ or Σ) and also ($\times$ or π);

(*c*) ϕ contains ($\times$ or π) but not ($+$ or Σ); and (*c*) is divided further into the types:

(*c*1) ϕ contains S;

(*c*2) ϕ does not contain S, but contains (π, $\bar{\pi}$);

(*c*3) ϕ does not contain S, but contains one of π, $\bar{\pi}$ only;

(*c*4) ϕ does not contain S, π or $\bar{\pi}$.

Case (*a*). By § 2.2, if ϕ contains $\mp$, $\bar{\Sigma}$, $\overline{\times}$, $\bar{\pi}$ or P a multiplicative system with units need not have canonical extension to ϕ. So we have only to consider ϕ for which ($+$ or Σ) $\varepsilon\, \phi \subset (0, 1, +, \Sigma, S)$. When $\Sigma\, \varepsilon\, \phi$, cases (2) and (3′) of part (i) of Theorem 3 cover all cases except when $S\, \varepsilon\, \phi \subset (0, +, \Sigma, S)$. This is covered by §2.4.

Case (*b*). The examples of § 4.1 and § 4.2 prove that an extension to ϕ may not exist even for a modular lattice with units if ϕ contains one of the operations $\mp, \bar{\Sigma}, \overline{\times}, \bar{\pi}, P$ or S besides ($+$ or Σ) and ($\times$ or π). Otherwise $\phi \subset (0, 1, +, \Sigma, \times, \pi)$. The cases (1), (2) of part (i) above cover all such ϕ except when $+\, \varepsilon\, \phi \subset (0, 1, +, \times)$. This case comes under the example of §2.1 which proves the non-existence of a canonical extension to such a ϕ also.

Case (*c*). (*c*1). By §4.2 if ($\times$ or π) and S are in ϕ even a modular lattice with units may have no (canonical) extension to ϕ.

(*c*2). By §4.3 there is no canonical extension of even a distributive lattice with units if ϕ contains with (π, $\bar{\pi}$) also ($\mp$ or $\bar{\Sigma}$). Otherwise (π, $\bar{\pi}$) are in ϕ and $\phi \subset (0, 1, \times, \pi, \overline{\times}, \bar{\pi}, P)$, in which case § 3 proves the non-existence of a canonical extension to ϕ even for a modular lattice with units.

($c3$). $\pi \varepsilon \phi$; $\bar{\pi}$, S *are not in* ϕ ; so that $\pi \varepsilon \phi \subset (0, 1, \times, \pi, \mp, \bar{\Sigma}, \bar{\times}, P)$. When $\bar{\Sigma} \varepsilon \phi$, by Theorem 1 of § 1, there is a canonical extension to ϕ of any multiplicative system with units. If $\bar{\Sigma}$ is not in ϕ, but $\mp \varepsilon \phi$, we get one of the unresolved cases, coming under type (6). While if $\mp$ also is not in ϕ, § 3 gives non-existence if $\bar{\times} \varepsilon \phi$. Otherwise $\pi \varepsilon \phi \subset (0, 1, \times, \pi, P)$. In such a case there are canonical extensions when P is not in ϕ and when $(P, 0)$ are in ϕ besides π; finally if $P \varepsilon \phi \subset (1, \times, \pi, P)$, § 2.4 gives non-existence again.

$\bar{\pi} \varepsilon \phi$; π, S *are not in* ϕ ; so that $(\times, \bar{\pi}) \varepsilon \phi \subset (0, 1, \times, \mp, \bar{\Sigma}, \bar{\times}, \bar{\pi}, P)$. By §2.3 there is no canonical extension if ϕ does contain $\times$; § 3 gives non-existence if ϕ does not contain $\mp$ or $\bar{\Sigma}$ (since $\bar{\pi} \varepsilon \phi$). While if ϕ contains $\times$, $\mp$ or $\bar{\Sigma}$ together with $\bar{\pi}$, we get one of the unresolved cases, case 7.

($c4$). Now $\times \varepsilon \phi \subset (0, 1, \times, \mp, \bar{\Sigma}, \bar{\times}, P)$. If $\bar{\Sigma} \varepsilon \phi$ we get an existence case [(i), (4)] by Theorem 1. If $\bar{\Sigma}$ and P are not in ϕ, there exist canonical extensions when $\mp \varepsilon \phi$, or when $\mp$, $\bar{\times}$ are not in ϕ ; while if $\bar{\times} \varepsilon \phi \subset (0, 1, \bar{\times}, P)$ there is no canonical extension by § 3.

And when P and $\mp$ are in $\phi \subset (0, 1, \times \mp, \bar{\times}, P)$ we have one of the unresolved cases, (case 6). This completes the ϕ under ($c4$) ; and the proof of part (ii) of Theorem 3.

PROOF OF THEOREM 4. By using the fact that the dual of a lattice or a modular lattice, with units, is also such a lattice, and the duality theorem of E II, we can deduce part (i) from part (i) of Theorem 3.

Again all the arguments under cases (b), (c) of part (ii) of Theorem 3 hold also for lattices and modular lattices, except the case of a ϕ for which $+ \varepsilon \phi \subset (0, 1, +, \times)$. In this case, the lattice or modular lattice being closed for ϕ is its own canonical extension to ϕ. By the duality theorem applied to the cases under (c), we get the cases under (a), except the duals of cases under (6) and (7). So the proof of part (ii) is complete.

5.2. *The unresolved cases.* There are just two types for ϕ when the existence question is not decided by the earlier examples and constructions, namely the cases (6) and (7).*

But these two cases also can be resolved when the given multiplicative system with units is finite.

For when K is a finite multiplicative system with units, then all the extensions, by comprincipal ideals (or Σ-ideals), by $c\mathcal{N}$-ideals, and by Σ-ideals are also finite, consisting as they do of certain subsets of a finite set. Hence if L is the extension by $\bar{\Sigma}$-ideals, its elements have representations as finite distributive sums of the elements in the image of K; that is, the two extensions of Theorems 1 and 2 coincide; also L is closed for $\bar{\pi}$, as it is closed for $\mp$. Hence L is the extension to ϕ, if $(\times, \mp) \varepsilon \phi \subset (0, 1, \times, \pi, \mp, \bar{\Sigma}, \bar{\times}, \bar{\pi}, P)$. The types of ϕ under (6) and (7) all come under this description except when $(\mp, P) \varepsilon \phi \subset (0, 1, \bar{\times}, \mp, P)$. But then we can appeal to the example of §2.3 which proves that a canonical extension need not exist for such a ϕ. Thus all choice of ϕ under (6) or (7) can be covered if the given multiplicative system is finite. Similarly if K is a finite lattice or modular lattice with units, the dual results to the above are also true, and all ϕ coming under (6), (7) or (6′), (7′) can be dealt with.

* The duals of these, when the given ordered set is a lattice or a modular lattice with units admit of definitive solution if (6) or (7) is solved, by the duality theorem.

THE MAXIMUM TERM OF AN ENTIRE SERIES (V)

BY

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1. INTRODUCTION. Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of order ρ and lower order λ and let $\mu(r)$ denote the maximum term of the series for $|z| = r$, $\nu(r)$ its rank and R_n the 'sprungstellen' of $\nu(r)$ so that when $R_n < R_{n+1}$, $\nu(r) = n$ for $R_n \leqslant r < R_{n+1}$. Let

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, n \log \frac{R_{n+1}}{R_n} = \frac{b}{a};\ \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, \frac{\log \mu(r)}{\nu(r)} = \frac{d}{c}$$

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, \frac{\log |a_n/a_{n+1}|}{\log n} = \frac{\beta}{\alpha};\ \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, \frac{-\log |a_n|}{n \log n} = \frac{B}{A}.$$

I have proved elsewhere that [1], [2].

$$\alpha \leqslant A = \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant B \leqslant \beta \tag{1}$$

$$c \leqslant \frac{1}{\rho} \leqslant \frac{1}{\gamma} \leqslant d. \tag{2}$$

Polya and Szego have shown that [3, p. 13]

If $a_n > 0$ and if

$$\lim_{n\to\infty} n \left\{ \frac{a_n^2}{a_{n-1}\, a_{n+1}} - 1 \right\} = \frac{1}{k} \quad (0 < k < \infty) \tag{3}$$

then $f(z) = \sum_0^\infty a_n z^n$ is an entire function of order k and $\log \mu(r) \sim \nu(r)/k$.

2. STATEMENT OF RESULTS. Let

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, n \left\{ \frac{|a_n|^2}{|a_{n-1}\, a_{n+1}|} - 1 \right\} = \frac{L}{l}.$$

It is easy to show that

If $l > 0$, $f(z) = \sum a_n z^n$ is an entire function; but the converse is not true, since for any gap series $l = \alpha = -\infty$, $\beta = L = \infty$.

I prove here (4)

THEOREM 1. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an entire function, then*

$$l \leqslant \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log |a_n/a_{n+1}|}{\log n} \leqslant L \tag{5}$$

$$\liminf_{n\to\infty} n \log \frac{R_{n+1}}{R_n} \leqslant \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(r)}{\nu(r)} \leqslant \limsup_{n\to\infty} n \log \frac{R_{n+1}}{R_n} \tag{6}$$

$$\liminf_{n\to\infty} n \log \frac{R_{n+1}}{R_n} = \text{Max } (0, l) \tag{7}$$

$$\limsup_{n\to\infty} n \log \frac{R_{n+1}}{R_n} \leqslant L. \tag{8}$$

THEOREM 2. *If* $\log \mu(r)/\nu(r)$ *tends to a limit* d $(0 \leqslant d \leqslant \infty)$ *as* r *tends to infinity, and if*

$$\frac{R_{n+1}^n R_{n-1}^{n-1}}{R_n^{2n-1}} = 1 + O\left(\frac{1}{n}\right)$$

then

$$\lim_{n\to\infty} n \log \frac{R_{n+1}}{R_n} = d.$$

3. REMARKS. From (2), (6), (7) and (8) I obtain the following generalization of (3).

$$l \leqslant a \leqslant c \leqslant \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant d \leqslant b \leqslant L \tag{9}$$

and* from (1) and (5) I have

$$l \leqslant \alpha \leqslant A = \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant B \leqslant \beta \leqslant L. \tag{10}$$

* If $|a_n/a_{n+1}| > K |a_{n-1}/a_n|$ for $n > n_0$, where $K > 1$, then $l = L = \infty$ and from (9) we have $\rho = \lambda = 0$ Hence the statement of Theorem 5 of my paper [1, p. 1048] should be modified as follows. The proof remains the same.

THEOREM 5. *Let* $f(z) = \sum_0^\infty a_n z^n$ *be an entire function and let* $m(r,f) = \min_{|z|=r} |f(z)|$, $|\varepsilon_n| = 1$ *or* 0, $\varepsilon_n \neq 0$ *for an infinity of* n, $\vartheta = 2.2$. *If*

$|a_n/a_{n+1}| \geqslant \vartheta^2 |a_{n-1}/a_n|$ *for* $n > n_0$ *and if* $F(z) = \sum_0^\infty a_n \varepsilon_n z^n$ *then*

$$m(r,f) \geqslant k M(r,f)\,; \quad m(r, F) \geqslant k M(r,f),$$

where k *is a positive constant, for a sequence of values of* $r \to \infty$.

For $f(z) = \exp(z)$ we have $l = L = 1$ and for $f(z) = \exp(z^2) + \exp(z)$ we have $l = \alpha = -\infty$, $L = \beta = \infty$, $A = c = d = \frac{1}{2}$, $a = 0$, $b = B = 1$. An analogy with a known result $\rho = \text{Max}(q, \rho_1)$ may lead us to suppose that $L = \text{Max}(\beta, b)$. This is false, since for the function $f(z)$ defined below, $L = 3$, $b = 2$, $\beta = 1$.

EXAMPLE.

$$f(z) = \sum_{1}^{\infty} \left(\frac{z^{2n}}{(2n)!} + \frac{z^{2n+1}}{(2n)!\,(2n+2)} \right).$$

$L = 3$, $R_{2n-1} = R_{2n} = \sqrt{2n(2n-1)}$, $b = 2$, $\beta - 1$, $a = 0$, $\alpha = 1$.

If $|a_n/a_{n+1}|$ is a non decreasing function of n for $n > n_0$, then we have $l = a$, $L = b$ and [1, p. 1048] $\alpha = A = 1/\rho$; $B = \beta = 1/\lambda$.

4. PROOF OF THEOREM 1. Let

$$\lim_{n\to\infty} {\sup \atop \inf} n \log \left| \frac{a_n^2}{a_{n-1}\, a_{n+1}} \right| = {L_1 \atop l_1}.$$

It follows easily that $L = L_1$, $l = l_1$.

(i) We suppose first that $l \neq \infty$, $l \neq -\infty$. We have

$$\left| \frac{a_n^2}{a_{n-1}\, a_{n+1}} \right| > \exp\left(\frac{l-\varepsilon}{n}\right) \quad (n \geqslant N).$$

Hence

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| K \frac{a_{N+1}^2}{a_N\, a_{N+2}} \cdots \frac{a_n^2}{a_{n-1}\, a_{n+1}} \right|,$$

$$\log \left| \frac{a_n}{a_{n+1}} \right| > \log |K| + (l-\varepsilon) \sum_{p=N+1}^{n} \frac{1}{p} \sim (l-\varepsilon) \log n.$$

Hence $\alpha \geqslant l$ which obviously holds if $l = -\infty$. If $l = \infty$ the argument gives that $\alpha = \infty$. Similarly $\beta \leqslant L$.

(ii) We may suppose that $|f(0)| = |a_0| = 1$. We also suppose $b < \infty$. If $R_n < R_{n+1}$ we have for $R_n \leqslant r < R_{n+1}$

$$\nu(r) = n, \; \mu(r) = \frac{r^n}{R_1 R_2 \ldots R_n}$$

$$\frac{\log \mu(r)}{\nu(r)} = \log \frac{r}{R_n}$$
$$+ \frac{(n-1)\log(R_n/R_{n-1}) + \ldots + \log(R_2/R_1)}{n}$$
$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

Now

$$\frac{R_{n+1}}{R_n} < \exp\left(\frac{b+\varepsilon}{n}\right) \quad (n > n_0).$$

Hence

$$\log \frac{r}{R_n} \leqslant \log \frac{R_{n+1}}{R_n} = o(1)\,; \limsup_{n\to\infty} \Sigma_2 \leqslant b.$$

Hence $d \leqslant b$ which obviously holds when $b = \infty$. Similarly $c \geqslant a$.

(iii) Either

$$\left|\frac{a_n}{a_{n+1}}\right| > \left|\frac{a_{n-1}}{a_n}\right| \text{ for all } n > n_0$$

in which case, for all large n

$$n \log \frac{R_{n+1}}{R_n} = n \log \left|\frac{a_n^2}{a_{n-1}a_{n+1}}\right|.$$

$$\liminf_{n\to\infty} n \log \frac{R_{n+1}}{R_n} = l \geqslant 0, \tag{11}$$

or

$$\left|\frac{a_n}{a_{n+1}}\right| \leqslant \left|\frac{a_{n-1}}{a_n}\right| \tag{12}$$

for an infinity of n, say $n = m_1, m_2, \ldots\ldots$

Then $|a_n| r^n$ $(n = m_1, m_2, \ldots)$ does not become the maximum term for any value of r and $R_{n+1} = R_n$ infinitely often. Hence from (12) we have

$$\liminf n \log \frac{R_{n+1}}{R_n} = 0 \geqslant l. \tag{13}$$

From (11) and (13) we have $a = \text{Max}(0, l)$.

(iv) We may suppose $L < \infty$. Hence $a_n \neq 0$ $(n > n_0)$. Let $|a_m| r^m$, $|a_n| r^n$, $|a_p| r^p$ be the consecutive maximum terms $(m \leqslant n-1 \leqslant p-2)$. Then $R_{n+1} = R_{n+2} = \ldots = R_p$; $R_{m+1} = R_{m+2} = \ldots = R_n$. Further

$$R_n = \text{Max}\left\{ \left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{\frac{1}{2}}, \ldots, \left|\frac{a_m}{a_n}\right|^{\frac{1}{n-m}}, \ldots \right\}$$
$$= |a_m/a_n|^{\frac{1}{n-m}}, \qquad (14)$$

since the maximum term goes from $|a_m| r^m$ to $|a_n| r^n$. Similarly

$$R_p = |a_n/a_p|^{\frac{1}{p-n}} = \text{Max}\left\{ \left|\frac{a_{p-1}}{a_p}\right|, \ldots\ldots \left|\frac{a_{n+1}}{a_p}\right|^{\frac{1}{p-n-1}}, \ldots \right\}. \qquad (15)$$

Hence

$$n \log \frac{R_{n+1}}{R_n} = n \log \frac{R_p}{R_{m+1}} = n \log \frac{|a_n/a_p|^{\frac{1}{p-n}}}{|a_m/a_n|^{\frac{1}{n-m}}}.$$

Now from (14) and (15)

$$\log \frac{1}{|a_m|} \leqslant -(n-m) \log |a_{n-1}| - (1-n-m) \log |a_n|$$

$$\log \frac{1}{|a_p|} \leqslant (p-n-1) \log |a_n| - (p-n) \log |a_{n+1}|.$$

Hence

$$n \log \frac{R_{n+1}}{R_n} \leqslant n \log \left| \frac{a_n^2}{a_{n-1}\, a_{n+1}} \right| < L+\varepsilon \quad (n > n_0).$$

Let $m < \theta \leqslant n$ $(m > n_0)$. If $\theta = m+1, \ldots, n-1$, then

$$\theta \log \frac{R_{\theta+1}}{R_\theta} = 0. \quad \text{Hence } b \leqslant L.$$

5. PROOF OF THEOREM 2. This follows from Hardy's theorem, in Cesaro Summability, that if

$$\frac{s_1+s_2+\ldots\ldots+s_n}{n} \to d \ (0 \leqslant d \leqslant \infty) \text{ as } n \to \infty \text{ and if}$$

$$s_n - s_{n-1} = O\left(\frac{1}{n}\right) \text{ then } \lim_{n\to\infty} s_n = d.$$

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ON VON STERNECK-RAMANUJAN FUNCTION

BY

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1. INTRODUCTION. Bachmann, in *Niedere Zahlentheorie*, Bd. 2, 230-41, defines Von Sterneck's function $f(M, N)$ as the excess of the number of partitions of N into an even number of distinct parts (mod M), over the number of those into an odd number (zero, not being admitted as a part) and shows that $f(M, N) = \mu(M/g) \times \phi(M)/\phi(M/g)$, where $\mu(M)$ is Möbius function, $\phi(M)$ Euler's function and $g = (M, N)$, the g.c d. of M and N. Some years ago, K. G. Ramanathan(1) pointed out the remarkable fact that Von Sterneck's function is identical with Ramanujan's function $C_M(N)$.* It may be mentioned here that Aurel Wintner(2) had also stated before that

$$C_M(N) = \mu\left(\frac{M}{g}\right) \frac{\phi(M)}{\phi(M/g)}; \qquad (1.1)$$

but he does not seem to have identified Ramanujan's function with that of Von Sterneck. Since the two functions are identical, it is appropriate to refer to $f(M, N)$, or, $C_M(N)$ as *Von Sterneck-Ramanujan function.* We shall however write $C_M(N)$ only to denote the function.

Ramanujan(3) proved that

$$C_M(N) = \sum_{d/M, N} \mu\left(\frac{M}{d}\right) d = \sum_{d/g} \mu\left(\frac{M}{d}\right) d \qquad (1.2)$$

and Hardy(4) showed that $C_M(N)$ is multiplicative in M; while I have pointed out(5) that $C_M(N)$ is a multiplicative function of both M and N, with the modulus M and that

* Vide, *Collected papers of Srinivasa Ramanujan,* (Cambridge, 1927), 179-80.

(1.2) represents the 'identical equation' of $C_M(N)$ and can be written as

$$C_M(N) = E^{-1}(M)\ E(N).\ \text{princ I}. \qquad (1.3)$$

Several preperties of Von Sterneck-Ramanujan function were derived(5), (6) from the fact that it is multiplicative in both the arguments* and also by using (1.3). The object of the present paper is to es ablish some further properties of this function.

2. PROPERTIES OF VON STERNECK-RAMANUJAN FUNCTION.

THEOREM I. $C_M(N) = C_{M/d}\left(\frac{N}{d}\right)\phi(M)/\phi(M/d)$, *where d is any common divisor of M and N.*†

For, let

$\left(\frac{M}{d}, \frac{N}{d}\right) = \delta$; $(M, N) = g$. Then clearly $\delta = \frac{g}{d}$.

Now from (1.1)

$$C_M(N) = \mu\left(\frac{M}{g}\right)\frac{\phi(M)}{\phi(M/g)}$$

$$= \mu\left(\frac{M/d}{g/d}\right)\frac{\phi(M)}{\phi\left(\frac{M}{d}/g/d\right)}\left\{\frac{\phi(M/d)}{\phi(M/d)}\right\}$$

$$= \left\{\mu\left(\frac{M}{d/\delta}\right)\frac{\phi(M/d)}{\phi(M/d/\delta)}\right\}\frac{\phi(M)}{\phi(M/d)}.$$

* In *Math. Reviews*, 9, (1948), the reviewer of [6] remarks that the properties proved there, can be derived independently of the ideas of the *Theory of Multiplicative functions* The author wishes to express the fact that in the light of the *Theory of Multiplicative functions*, one discovers readily many properties which are not seen *easily* otherwise and that after we know that the function has *these properties*, it may be possible in many cases to give proofs not based on *Multiplicative function Theory* While the author was aware of what the reviewer has pointed out, viz proofs by changing summation orders, he employed different methods (sometimes longer) in [6], as he was adopting another point of view.

† In an unpublished paper, which M. V Subba Rao kindly showed me, he has obtained the same result by a different method.

Hence $C_M(N) = C_{M/d}(N/d)\ \phi(M)/\phi(M/d)$.

COR. (i) $C_M(N)\ d(g) = \phi(M) \sum_{d/g} C_{M/d}\left(\frac{N}{d}\right)/\phi(M/d)$,

where $d(g)$ represents the number of divisors of g.

(ii) $C_M(N) \sum_{d/g} \phi\left(\frac{M}{d}\right) = \phi(M) \sum_{d/g} C_{M/d}(N/d)$.

THEOREM 2. $\sum_{d/g} C_d(d) = g$, *where g is the greatest common divisor of M, N.*

For, $$C_d(d) = \mu\left(\frac{d}{d}\right)\frac{\phi(d)}{\phi(d/d)} = \phi(d) \text{ from (1.1).}$$

Hence $\sum_{d/g} C_d(d) = \sum_{d/g} \phi(d) = g$.

THEOREM 3. $\Sigma C_M\ (Nh) = g\phi(M)\ E_0\left(\frac{M}{g}\right)$, *where h runs through a complete set of residues (mod M).*

To prove the theorem, we will use the following

LEMMA. *If $(a, m) = g$, $a = g\,a'$, $m = gm'$ and $g|K$, the remainders of the series K, $K+a, \ldots\ldots, K+(m-1)a$, when divided by m, are the numbers $0\,g$, $1\,g$, $2\,g, \ldots, (m-1)\,g$, continually repeated in a certain order.**

PROOF. From (1.1), it follows that

$$C_M(N) = C_M(g) \tag{2.1}$$

and that $C_M(N) = C_M(N + \lambda M)$, where λ is any integer. (2.2)

Therefore, from the lemma, by writing $m = M$, $a = K = N$ and using (2.1) and (2.2) we see that

$\Sigma C_M(Nh) = \sum_{h=1}^{M} C_M(Nh) = g\Sigma C_M(gr)$, where r runs through a complete set of residues (mod M/g).

Now $(M, gr) = g(M/g, r) = g\delta$ if $(M/g, r) = \delta$.

* Vide Chrystal, *Algebra*, Pt. II, 547-50.

Hence by (2.1), $C_M(gr) = C_M(g\delta)$. Arguing exactly as in Theorem (4.1.2) of [5], we have,

$$\Sigma\, C_M(gr) = \sum_{\delta/M/g} C_M(g\delta)\,\phi\left(\frac{M}{g\delta}\right)$$

$$= \sum_{\delta/M/d} \mu\left(\frac{M}{g\delta}\right) \frac{\phi(M)}{\phi(M/g\delta)}\,\phi\left(\frac{M}{g\delta}\right) \text{ using (1.1)}$$

$$= \phi(M) \sum_{\delta/M/g} \mu\left(\frac{M}{g\delta}\right)$$

$$= \phi(M)\; E_0(M/g).$$

Therefore

$$\Sigma\, C_M(Nh) = g\;\phi(M)\; E_0(M/g).$$

REMARK. Theorem 3 is more general than Theorem (4.1.2) of [5] which turns out to be a special case for $g = 1$.

THEOREM 4. $\Sigma\, C_M(N\delta) = \mu\left(\frac{M}{g}\right)\frac{\{\phi(M)\}^2}{\phi(M/g)}$, *where* δ *runs through all integers not greater than and prime to* M.

For if $(\delta, M) = 1$, then $(M, N\delta) = (M, N) = g$. Since there are $\phi(M)$ numbers not greater than and prime to M, we have by (2.1)

$$\Sigma\, C_M(N\delta) = C_M(g)\;\phi(M)$$

$$= \mu\left(\frac{M}{g}\right)\frac{\{\phi(M)\}^2}{\phi(M/g)} \text{ using (1.1).}$$

THEOREM 5. *If* $d|M$, *and* $\left(d, \frac{M}{d}\right) = 1$, *then* $\Sigma\, C_d(M/d) = 1$, *when* $M = 1$, *or every prime factor of* M *is repeated; and* $= 0$, *when* M *has any non-repeated prime factor.*

For, if $\psi(M) = \Sigma\; C_d(M/d)$, then by (2.1) since $(d, M/d) = 1$ we have $\psi(M) = \Sigma\; C_d(1) = \Sigma\;\; \mu(d)$ from (1.1).

Since $\psi(M)$ is clearly multiplicative, $\psi(1) = 1$ and it is sufficient if we calculate $\psi(M)$ when M is a power of a prime,

Let $M = p^m$. When $m = 1$, $\psi(p^m) = \psi(p) = \mu(1) + \mu(p) = 1 - 1 = 0$.

If $m > 1$, $\psi(p^m) = \mu(1) + \mu(p^m) = 1 + 0 = 1$.

Hence it follows that when $M = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$,

$$\psi(M) = 1 \text{ if } M = 1 \text{ or if } m_i > 1 \text{ for } i = 1, \ldots, r,$$
$$= 0 \text{ if any one } m_i = 1;$$

which is the same as the result stated in the theorem.

THEOREM 6. *$C_M(N) C_N(M) = \phi(g) C_l(g)$, where l is the least common multiple of M, N and $g = (M, N)$.*

The theorem follows from the

LEMMA. *If $F(M, N)$ is a multiplicative function of M N and is symmetric, i.e. $F(M, N) = F(N, M)$, then $F(M, N) = F(l, g)$.*

To prove the lemma, we note that since $F(M, N)$ is multiplicative $F(M, N) = \prod_{i=1}^{r} F(p_i^{\alpha_i}, \; p_i^{\beta_i})$, where $M = \prod_{i=1}^{r} p_i^{\alpha_i}$; $N = \prod_{i=1}^{r} p_i^{\beta_i}$ $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $i = 1, \ldots, r$.

On account of the symmetry of $F(M, N)$ we can write

$$F(p_i^{\alpha_i}, p_i^{\beta_i}) = F(p_i^{\beta_i}, p_i^{\alpha_i}) \text{ if } \beta_i > \alpha_i.$$

Therefore $F(M, N) = \prod_{i=1}^{r} F(p_i^{\gamma_i}, p_i^{\delta_i})$,

where γ_i = greater of α_i, β_i and δ_i = smaller of α_i, β_i.

Hence $F(M, N) = F(l, g)$ since $l = \prod_{i=1}^{r} p_i^{\gamma_i}$ and $g = \prod_{i=1}^{r} p_i^{\delta_i}$.

Since the product $C_M(N) C_N(M)$ is a symmetric multiplicative function, it follows from the lemma that

$$C_M(N) C_N(M) = C_l(g) C_g(l) = C_l(g) C_g(g) \text{ by } (2.1)$$
$$= C_l(g) \phi(g) \text{ by } (1.1).$$

3. RELATION BETWEEN VON STERNECK-RAMANUJAN FUNCTION AND KRONECKER'S FUNCTION.(7)

Kronecker's function $\rho(M, N)$ is defined as

$\rho(M, N) = 1$ if $(M, N) = 1$ and $= 0$ otherwise. It is shown(6) that $\rho(M, N) = E_0(g)$ and is multiplicative in M, N with the identical equation

$$\rho(M, N) = E(M)\, E(N).\ \text{princ}\ \mu(M). \qquad (3.1)$$

Some properties of $\rho(M, N)$ were derived before,* using (3.1). Before pointing out the connection between Von Sterneck-Ramanujan function and Kronecker's function, it is worthwhile to mention some more properties of $\rho(M, N)$.

From the definition of $\rho(M, N)$ we get

$$\sum_{d|g} \rho(d, d) = 1 = \sum_{d|g} \rho(M/d, N/d). \qquad (3.2)$$

If $d|M$, $(d, M/d) = 1$, then $\Sigma\, \rho(d, M/d) =$ the number of block factors† of $M = 2^k$, where k is the number of distinct prime factors of M.

THEOREM 7. $\sum_{d|M} \rho(d, N) = \sum_{\delta|g} d\left(\frac{M}{d}\right)\mu(\delta)$, *where* $d(M)$ *denotes the number of divisors of* M *and* $g = (M, N)$.

For,

$$\begin{aligned}\sum_{d|M} \rho(d, N) &= \rho(M, N).E(M)\, E_0(N)\\ &= E(M)\, E(N).\text{princ}\ \mu(M).E(M)\, E_0(N) \text{ by } (3.1)\\ &= \{E(M)\, E(N)\}.\{E(M)\, E_0(N)\}.\text{princ}\ \mu(M)\\ &= \{E(M).E(M)\}\ \{E(N).E_0(N)\}.\text{princ}\ \mu(M)\\ &= d(M)\, E(N).\text{princ}\ \mu(M)\\ &= \sum_{\delta/g} d\left(\frac{M}{\delta}\right)\mu(\delta).\end{aligned}$$

REMARK. Since $\rho(M, N) = 0$ if $(M, N) \neq 1$, the summation on the left is only for divisors of M which are prime to N.

THEOREM 8. $\sum_{d|N} \rho(M, d) = \sum_{\delta|g} d\left(\frac{N}{\delta}\right)\mu(\delta)$.

* Vide [6], pp. 59-61.

† The result is substantially the same as Th. 6 of [6].

The proof is on the same lines as in Theorem 7, for

$$\sum_{d\mid N} \rho(M, d) = \rho(M, N).E_0(M)\,E(N).$$

NOTE. The summation on the left is clearly over only those divisors of N, prime to M.

THEOREM 9. *If* $d_r \mid M$, $\delta_s \mid N$,

$$\Sigma\,\Sigma\; \rho(d_r, \delta_s) = \sum_{\delta \mid g} d\left(\frac{M}{\delta}\right) d\left(\frac{N}{\delta}\right)\mu(\delta).$$

Now

$$\sum_{d_r \mid M}\sum_{\delta_s \mid N} \rho(d_r, \delta_s) = \rho(M, N).E(M)\,E(N).$$

Hence proceeding similarly as in Theorem 7, the result follows.

REMARK. It may be easily seen that only divisors for which $(d_r, \delta_s) = 1$, need be taken on the left side.

4. RELATIONSHIP BETWEEN $C_M(N)$ and $\rho(M, N)$.

THEOREM 10. *The composite of Von Sterneck-Ramanujan function with Kronecker's function is an ordinal function with N as the major argument which is equal to $d(N/M)\ \phi(M)$ whenever M divides N.*

For

$$\begin{aligned} C_M(N).\rho(M,N) &= \{E^{-1}(M)\,E(N).\text{princ I}\}.\{E(M)\,E(N).\\ &\qquad \text{princ}\ \mu(M)\}\ \text{by (1.3) and (3.1)}\\ &= E^{-1}(M)\,E(N).E(M)\,E(N).\text{princ I}.\\ &\qquad\qquad \text{princ}\ E^{-1}\\ &= E_0(M)\,d(N).\text{princ}\ (\text{I}.E^{-1})\\ &= E_0(M)\,d(N).\text{princ}\ \phi(M)\\ &= d\left(\frac{N}{M}\right)\phi(M)\ \text{if}\ M \mid N\ \text{and 0 otherwise.} \end{aligned}$$

NOTE. Since

$$C_M(N).\rho(M,N) = \sum_{d\mid M}\sum_{\delta\mid N} C_{M/d}\left(\frac{N}{\delta}\right)\rho(d, \delta),$$

and $\rho(M, N) = 1$ if $(M, N) = 1$ and 0 otherwise, we have

$$\Sigma \Sigma C_{M/d}\left(\frac{N}{\delta}\right) = d\left(\frac{N}{M}\right)\phi(M) \text{ if } M \mid N$$
$$= 0, \text{ otherwise}$$

the summation on the left being for such divisors d of M and δ of N which are mutually prime, i.e. $(d, \delta) = 1$.

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NOTE ON CRITIC CENTRES AND COINCIDENCE POINTS OF CUBIC CURVES

BY

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1. It has been shown by Hart that the 72 coincidence points of a bicursal cubic Γ (i.e. a curve of deficiency one) given in its canonical form

$$x^3+y^3+z^3+6\,mxyz=0$$

are its intersections with the four pairs of cubics (S, T), (S_r, T_r), $r=1, 2, 3$ defined by

$$S\equiv x^3+\omega\, y^3+\omega^2\, z^3=0$$
$$T\equiv x^3+\omega^2\, y^3+\omega\, z^3=0$$
$$S_1\equiv x^2y+y^2\, z+z^2\, x=0$$
$$T_1\equiv xy^2+yz^2+zx^2=0$$
$$S_2\equiv x^2y+\omega y^2z+\omega^2z^2x=0$$
$$T_2\equiv xy^2+\omega^2yz^2+\omega zx^2=0$$
$$S_3\equiv x^2y+\omega^2y^2z+\omega z^2x=0$$
$$T_3\equiv xy^2+\omega yz^2+\omega^2zx^2=0,$$

where ω is an imaginary cube root of unity. The purpose of this paper is to investigate the geometrical relations that subsist between the four pairs of subsidiary cubics given above and the given cubic Γ.

2. Defining, as in the case of a conic, a self-conjugate triangle of a plane cubic Γ as a triangle, each of whose sides is the polar line of the opposite vertex with respect to Γ, it is evident that *the triangle of reference is self-conjugate to any cubic of the type*

$$ax^3+by^3+cz^3=0. \qquad (1)$$

Every such cubic is equianharmonic, i.e. the pencil of four tangents that can be drawn to the cubic from any point on it is equianharmonic.

* *Transactions of the Royal Irish Academy*, Vol. XXV.

The cubics S and T therefore possess these properties.

LEMMA. *The triangle of reference ABC is a "Hart-triangle"* for any cubic of the type*

$$lx^2y+my^2z+nz^2x=0. \qquad (2)$$

This follows because the tangentials of A, B, C w.r.t. (2) are respectively C, A, B, so that each vertex coincides with its third tangential and is accordingly a coincidence point.

The same property holds good with respect to the cubic

$$l'xy^2+m'yz^2+n'zx^2=0. \qquad (3)$$

While the cyclic substitution (ACB) determines the tangentials of A, B, C for the cubic (2), the substitution (ABC) appertains to the cubic (3).

3. The coordinates of the twelve critic centres† of Γ are given by

$$\begin{array}{lll} A(1,0,0), & B(0,1,0), & C(0,0,0), \\ A_1(1,1,1), & B_1(1,\omega^2,\omega), & C_1(1,\omega,\omega^2), \\ A_2(\omega^2,1,1), & B_2(1,\omega^2,1), & C_2(1,1,\omega^2), \\ A_3(\omega,1,1), & B_3(1,\omega,1), & C_3(1,1,\omega). \end{array}$$

Let Δ, Δ_r, denote the triangles ABC, $A_rB_rC_r$ $(r=1,2,3)$. The sides of these four triangles are the twelve lines that can be drawn through the nine inflexions of Γ.

If (X_r, Y_r, Z_r) be the coordinates, when referred to the triangle Δ_r, of any point (x, y, z) referred to Δ, we have the transformations

$$\begin{array}{l|l|l} x=X_1+\omega^2Y_1+\omega Z_1 & x=\omega^2X_2+Y_2+Z_2 & x=\omega X_3+Y_3+Z_3 \\ y=X_1+\omega Y_1+\omega^2Z_1 & y=X_2+\omega^2Y_2+Z_2 & y=X_3+\omega Y_3+Z_3 \\ z=X_1+Y_1+Z_1 & z=X_2+Y_2+\omega^2Z_2 & z=X_3+Y_3+\omega Z_3. \\ \quad (a) & \quad (b) & \quad (c) \end{array}$$

These transformations transform S and T as follows:

$$S\equiv x^3+\omega y^3+\omega^2z^3=9\omega^2(X_1^2Y_1+Y_1^2Z_1+Z_1^2X_1)$$

* Hilton: *Plane Algebraic curves*, p 236. Ex. 2 and p 260 Ex. 15.

† Hilton: *l.c.* p. 235.

$$= 9(\omega X_2^2 Z_2 + Z_2^2 Y_2 + \omega^2 Y_2^2 X_2) = 9(\omega^2 X_3^2 Y_3 + Y_3^2 Z_3 + \omega Z_3^2 X_3),$$

$$T \equiv x^3 + \omega^2 y^3 + \omega z^3 = 9\omega (X_1 Y_1^2 + Y_1 Z_1^2 + Z_1 X_1^2)$$

$$= 9(\omega^2 X_2 Z_2^2 + Z_2 Y_2^2 + \omega Y_2 X_2^2) = 9(\omega X_3 Y_3^2 + Y_3 Z_3^2 + \omega^2 Z_3 X_3^2).$$

The equianharmonic cubics S and T have therefore each of the triangles Δ_1, Δ_2, Δ_3 as a Hart-triangle.

The cubics S_1, T_1 are transformed as follows:

$$S_1 \equiv x^2y + y^2z + z^2x = 3(X_1^3 + \omega^2 Y_1^3 + \omega Z_1^3)$$

$$= (3+6\omega^2)(X_2^2 Y_2 + Y_2^2 Z_2 + Z_2^2 X_2)$$

$$= (3+6\omega)(X_3^2 Y_3 + Y_3^2 Z_3 + Z_3^2 X_3),$$

$$T_1 \equiv xy^2 + yz^2 + zx^2 = 3(X_1^3 + \omega Y_1^3 + \omega^2 Z_1^3)$$

$$= (3+6\omega^2)(X_2 Y_2^2 + Y_2 Z_2^2 + Z_2 X_2^2)$$

$$= (3+6\omega)(X_3 Y_3^2 + Y_3 Z_3^2 + Z_3 X_3^2).$$

It follows that the cubics S_1, T_1 are also equianharmonic, having Δ_1 as a common self-conjugate triangle, and Δ, Δ_2, Δ_3 as common Hart-triangles. Similar properties are verified for the cubic pairs S_2, T_2 and S_3, T_3 which have respectively Δ_2, Δ_3 as common self-conjugate triangles, and the remaining three triangles as common Hart-triangles.

4. The number of plane cubics that can be drawn through nine points is either 1 or ∞; for example, through the nine inflexions of Γ there passes a pencil of cubics including the Hessian of Γ and the triangles Δ, Δ_1, Δ_2, Δ_3. The nine vertices of any three of these four triangles similarly determine a pencil of cubics which includes the two cubics having those three triangles as common Hart-triangles and the remaining one as a common self-conjugate triangle.

THE CHINESE RAMAINDER THEOREM FOR THE DISTRIBUTIVE LATTICE*

YB

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1. The Chinese Remainder Theorem† of classical number theory may be stated as follows :—

THEOREM 1. *If m, n are two relatively prime integers, then the congruence system*

$$x \equiv a \pmod{m}, \; x \equiv b \pmod{n}$$

has always a solution; further any two solutions x, y are such that $x \equiv y \pmod{m\,n}$.

In fact, this is only a particular case of the following more general theorem which may be legitimately called the general form of the Chinese Remainder Theorem :—

THEOREM 2. *If g, l denote respectively the g.c.d. and l.c.m. of m, n, then the congruences*

$$x \equiv a \pmod{m} \tag{1}$$
$$x \equiv b \pmod{n}$$

are soluble, if and only if, $a \equiv b \pmod{g}$; *further if x is a solution and* $y \equiv x \pmod{l}$, *y also is a solution ; conversely, any two solutions x, y satisfy the congruence* $x \equiv y \pmod{l}$.

PROOF. For, the congruences (1) imply that $(x-a)$, $(x-b)$ are multiples of m, n respectively. Hence $(x-a)$, $(x-b)$ are multiples of g, which shows that $a \equiv b \pmod{g}$ is a necessary condition for the coexistence of the congruences (1). For proving the sufficiency of this condition, we remark that, if $a \equiv b \pmod{g}$, then by a well-known result in the theory of indeterminate equations, there exist integers m_1, n_1 such that

$$mm_1 - nn_1 = b - a. \tag{2}$$

* I wish to thank Dr. R Vaidyanathaswamy for his help in the preparation of this note.

† Wright · *First course in Theory of Numbers*, New York, (1939), Theorem 16, p. 55.

If the simple continued fraction for m/n has r convergents with p/q as penultimate convergent, then one solution of (2) is the pair

$$m_1 = (-1)^r (b-a)q.$$
$$n_1 = (-1)^r (b-a)p.$$

The most general solution of (2) is evidently

$$m_1 = (-1)^r (b-a)\, q+\lambda n/g$$
$$n_1 = (-1)^r (b-a)\, p+\lambda m/g,$$

where λ is an arbitrary integer.

Now, if m_1, n_1 is any solution of (2) $x = mm_1 + a = nn_1+b$ is obviously a solution of (1). Next, suppose that x is a solution of (1) and $y \equiv x \pmod{l}$. Since $y-x \equiv 0 \pmod{m}$ and $x-a \equiv 0 \pmod{m}$, it follows that $y-a = y-x+x-a \equiv 0 \pmod{m}$. Similarly $y-b \equiv 0 \pmod{n}$, whence y also is a solution of (1). It remains to show that for any two solutions x, y of (1), $x \equiv y \pmod{l}$. This is an immediate consequence of the fact that $(x-y)$ being a multiple of m as well as of n, must be a multiple of l.

An alternative statement of the above theorem is:

THEOREM 2′. *If a, b are any two integers congruent* (mod g), *the residue-class* mod m *containing a intersects the residue-class* mod n *containing b in a definite residue-class* mod l.

For the particular case in which m, n are relatively prime (i.e. $g = 1$) Theorem 2 reduces to Theorem 1.

2. Our object in this note is to obtain the analogues of the above two theorems for a distributive lattice* L with 0. Guided by the definitions of elementary number theory, we can introduce the concept of the g.c.d. and the l.c.m. of two μ-ideals A_μ, B_μ of L. In the case of positive integers, the g.c.d. g of two integers m, n can be defined as that common divisor of m, n which is a multiple of every common divisor of m, n; similarly, the l.c.m. of two integers m, n is that common multiple l of m, n which is a

* For lattice-theoretic notions used in this note, vide R. Vaidyanathaswamy, *Treatise on set Topology*, Part 1 (1947), pp. 40-52.

divisor of every common multiple of m, n. Pursuing this analogy, since a divisor of a μ-ideal A_μ is a μ-ideal containing A_μ, we see that it is quite legitimate to define the g.c.d. and the l.c.m. of two ideals A_μ, B_μ as their ideal-sum $A_\mu+B_\mu$ and ideal-product $A_\mu.B_\mu$ respectively. Again, we say that $x\equiv y$ (mod A_μ), if and only if, there exist elements $s, t \varepsilon A_\mu$ such that $x+s=y+t$. It is easy to verify that this congruence is an equivalence relation partitioning L into disjoint residue-classes [(x, y) belonging to the same residue class, if and only if $x\equiv y$ (mod A_μ)] We are now in a position to state the analogue of Theorem 2.

THEOREM 3. *A necessary and sufficient condition that the congruence system*

$$x\equiv\alpha \text{ (mod } A_\mu)$$
$$x\equiv\beta \text{ (mod } B_\mu)$$

be soluble is that $\alpha\equiv\beta$ (mod $A_\mu+B_\mu$) ; *further, if* x *is a solution of* (1) *and* $y\equiv x$ (mod $A_\mu.B_\mu$) y *also is a solution of* (1) ; *conversely, any two solutions* x, y *satisfy the congruence* $x\equiv y$ (mod $A_\mu. B_\mu$).

PROOF. The 'necessity' follows from the obvious fact that given two ideals C_μ, D_μ such that $C_\mu\subset D_\mu$, $y\equiv z$ (mod C_μ) implies $y\equiv z$ (mod D_μ). Hence, x being any solution of system (1) we must have $x\equiv\alpha$ (mod $A_\mu+B_\mu$), $x\equiv\beta$ (mod $A_\mu+B_\mu$), giving $\alpha\equiv\beta$(mod $A_\mu+B_\mu$).

For proving the sufficiency, let us observe that since by hypothisis $\alpha\equiv\beta$ (mod $A_\mu+B_\mu$) there exists $t \epsilon A_\mu+B_\mu$ such that

$$\alpha+t=\beta+t.^* \tag{3}$$

Since $t \varepsilon A_\mu+B_\mu$ it is necessarily of the form

$$t=a+b,\ a \in A_\mu\ ;\ b \in B_\mu. \tag{4}$$

Now we shall show that $x_1=(a+\alpha)\ (b+\beta)$ is a solution of (1). Since by (2) and (3)

* *Ibid.*, p. 46.

$$x_1 \equiv x_1 + a \pmod{A_\mu} = (a+\alpha)\ (b+\beta)+a$$
$$= (a+\alpha)\ (a+b+\beta)$$
$$= (a+\alpha)\ (a+b+\alpha), \text{ by } (2)$$
$$= (a+\alpha) \equiv \alpha \pmod{A_\mu},$$

and similarly $x_1 \equiv \beta \pmod{B_\mu}$, it follows that x_1 is a solution of (1), as stated. Again, if x is a solution of (1) and $y \equiv x \pmod{A_\mu\ B_\mu}$, then y also is a solution of (1). For, $y \equiv x \pmod{A_\mu.B_\mu}$ implies the congruences $y \equiv x \pmod{A_\mu}$, $y \equiv x \pmod{B_\mu}$, whence y is also a solution. Finally, it remains to show that for any two solutions x, y of (1) $x \equiv y \pmod{A_\mu\ B_\mu}$. Since $x \equiv y$ (mod A_μ or B_μ), we must have $x+a_1 = y+a_1$; $a_1 \varepsilon A_\mu$; $x+b_1 = y+b_1$; $b_1 \varepsilon B_\mu$. Consequently, we have $x+a_1 b_1 = (x+a_1)\ (x+b_1) = (y+a_1)\ (y+b_1) = y+a_1 b_1$. Since $a_1\ b_1 \varepsilon A_\mu.B_\mu$ it follows that $x \equiv y \pmod{A_\mu.B_\mu}$.

Alternatively, we may state Theorem 3 as follows:

THEOREM 3′. *If α, β are any two elements of L mutually congruent w.r t. the ideal $A_\mu+B_\mu$, the residue-class* mod A_μ *containing α intersects the residue-class* mod B_μ *containing β in a definite residue-class* (mod $A_\mu\ B_\mu$).

In order to state a result for L which is analogous to Theorem 1, we must extend to the μ-ideals of L the notion of relative primality of integers. For the integers, there holds the theorem of unique factorization into primes, and two integers are relatively prime when they have no common prime factor. Analogously for the μ-ideals of L there is Stone's theorem* of unique factorization into prime ideals (even though an infinity of prime factors would occur and prime ideals are not necessarily divisorless). If we agree, that two μ-ideals A_μ, B_μ are to be

* Stone's factorization theorem for μ-ideals states that a μ-ideal (different from the whole lattice) is the product of all its prime μ-ideal-divisors Vide Stone. "Topological Representations of Distributive Lattices and Brouwerian Logics", *Cassopis pro pestovani Mathematisky A, Fysuky cast Mathematica*, Praha, (1937), Theorem 9.

called relatively prime if they have no common prime factors, then this would imply that they are not contained in any ideal other than the whole lattice, or, in other words, $A_\mu + B_\mu = 1$. Thus the lattice analogue of the restricted Chinese Remainder Theorem (Theorem 1) would run as follows:—

THEOREM 4. *If* $A_\mu + B_\mu = 1$ *and* α, β *are arbitrary elements of* L, *the congruence-system*

$$x \equiv \alpha \pmod{A_\mu}$$
$$x \equiv \beta \pmod{B_\mu}$$

has always a solution and any two solutions are mutually congruent w.r.t. the ideal $A_\mu . B_\mu$.

Since for any two elements α, β of L, $\alpha \equiv \beta \pmod{1}$, Theorem 4 follows as a particular case of Theorem 3.

FIVE FAMILIES OF RULED SURFACES THRO' A LINE OF A RECTILINEAR CONGRUENCE*

BY

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The object of this paper is to study the repeated applications of the operation of forming the Jacobian from Kummer's quadratic form.

Let the coordinates of a point on the surface of reference be given by x^i $(i = 1, 2, 3)$ and the direction cosines of the ray of the congruence be $\lambda^i (i = 1, 2, 3)$. Then the two quadratic forms used by Kummer[1] are

$$f \equiv G_{\alpha\beta}\, du^\alpha du^\beta \text{ and } \phi \equiv \mu_{\alpha\beta}\, du^\alpha du^\beta \;\; (\alpha, \beta = 1, 2),$$

where $\qquad G_{\alpha\beta} = \lambda^i_{,\alpha} \lambda^i_{,\beta}$ and $\mu_{\alpha\beta} = \lambda^i_{,\alpha}\, x^i_{,\beta}$,

$\lambda^i_{,\alpha}$ and $x^i_{,\alpha}$ denoting the covariant differentiation of λ^i and x^i respectively with respect to u^α.

(a) $f \equiv G_{\alpha\beta}\, du^\alpha du^\beta = 0$ denotes the equation of surfaces whose spherical representations are minimal lines, and (b) $\phi \equiv \mu_{\alpha\beta}\, du^\alpha du^\beta = 0$ is the equation of surfaces whose lines of striction lie on the surface of reference.

Now (c) $J(f, \phi) = 0$ is the same as

$$J(G_{\alpha\beta}\, du^\alpha du^\beta,\; \mu_{\alpha\beta}\, du^\alpha du^\beta) = 0$$

or

$$E^{\gamma\delta} G_{\alpha\gamma} (\mu_{\beta\delta} + \mu_{\delta\beta})\, du^\alpha du^\beta = 0, \qquad (1)$$

where

$$E_{\alpha\beta} = (\lambda^i_{,\alpha} \quad \lambda^i_{,\beta} \quad \lambda^i), \; E_{\alpha\beta} E_{\gamma\delta} = G_{\alpha\gamma} G_{\beta\delta} - G_{\alpha\delta} G_{\beta\gamma}$$

and $E^{\alpha\beta}$ is a tensor conjugate to $E_{\alpha\beta}$.

The equation (1) *is the equation of the Principal Ruled Surfaces.*

* I am thankful to Dr. Ram Behari for his guidance.

(d) If the two quadratic forms used by Sannia[2] are

$$f \equiv G_{\alpha\beta}\, du^\alpha\, du^\beta,\ \theta \equiv \xi_{\alpha\beta}\, du^\alpha\, du^\beta,$$

where $\xi_{\alpha\beta} = E^{\gamma\delta}\, G_{\alpha\gamma}\, \mu_{\delta\beta}$ (2)

or $\mu_{\delta\beta} = \xi_{\alpha\beta}\, E_{\gamma\delta}\, G^{\alpha\gamma}$ (3)

then $J[f, J(f, \phi)] = 0$ is the same as

$$J(G_{\alpha\beta}\, du^\alpha\, du^\beta,\ E^{\gamma\delta}\, G_{\alpha\gamma}\ (\mu_{\beta\delta} + \mu_{\delta\beta})\ du^\alpha\, du^\beta) = 0$$

or $E^{\varphi\psi}\, G_{\alpha\psi}\, E^{\gamma\delta}\, G_{\psi\gamma}\, (\mu_{\delta\beta} + \mu_{\beta\delta})\ du^\alpha\, du^\beta = 0$

or $E^{\gamma\delta}\, G_{\alpha\gamma}\, (\xi_{\beta\delta} + \xi_{\delta\beta})\ du^\alpha\, du^\beta = 0$ (4)

by virtue of (3).

The equation (4) . represents the surfaces of distribution.

NOTE :—Another form of the equation of the surfaces of distribution is obtained as follows :—

From the theory of quadratic forms,

$$J[f, J(f, \phi)] = 0 \text{ is the same as}$$

$$\frac{\phi}{f} = \frac{1}{2} \frac{F(f, \phi)}{\Delta f},$$

where $\frac{F(f, \phi)}{\Delta f} = \mu_{\alpha\beta}\, G^{\alpha\beta} = -(t_1 + t_2),$

$G^{\alpha\beta}$ being a tensor conjugate to $G_{\alpha\beta}$ and t_1 and t_2 being distances of limit points from the surface of reference.

Or $2\, \frac{\mu_{\alpha\beta}\, du^\alpha\, du^\beta}{G_{\alpha\beta}\, du^\alpha\, du^\beta} = \mu_{\alpha\beta}\, G^{\alpha\beta}$

or $t = \frac{1}{2}\, (t_1 + t_2),$ (5)

where t is the distance of the central point from the surface of reference.

Hence the surfaces of distribution are those whose lines of striction lie on the middle surface.

(e) Lastly, the surfaces $\mathcal{J}[\phi, \mathcal{J}(f, \phi)] = 0$ are given by

$$\frac{f}{\phi} = \frac{1}{2}\frac{F(f, \phi)}{\Delta\phi}$$

or

$$\frac{f}{\phi} = \frac{1}{2}\frac{F(f, \phi)}{\Delta f}\cdot\frac{\Delta f}{\Delta\phi},$$

where $\frac{\Delta\phi}{\Delta f} = \frac{1}{8} E^{\alpha\beta} E^{\gamma\delta} (\mu_{\alpha\gamma}+\mu_{\gamma\alpha})\ (\mu_{\beta\delta}+\mu_{\delta\beta}) = t_1\, t_2$

or

$$t = \frac{2t_1\, t_2}{t_1+t_2}.$$

Hence these are the surfaces for which the distance to the central point is given by the ratio of the product of the distances to the limit points and the distance from the middle point.

NOTE :—From the equation of the surfaces $\mathcal{J}[\phi, \mathcal{J}(f, \phi)] = 0$, we see that if $t_1+t_2 = 0$

then $\frac{1}{t} = 0$ or $G_{\alpha\beta}\, du^\alpha\, du^\beta = 0$.

But $t_1+t_2 = 0$ is the necessary and sufficient condition that the surface of reference is the middle surface. Hence, *these surfaces coincide with surfaces whose spherical representations are minimal lines, provided the surface of reference is the middle surface.*

Thus we get the result that the repeated applications of the operation of forming the Jacobian from Kummer's fundamental quadratic forms

$$f \equiv G_{\alpha\beta}\, du^\alpha\, du^\beta,\ \phi \equiv \mu_{\alpha\beta}\, du^\alpha\, du^\beta$$

lead to only five distinct families of ruled surfaces of the congruence, viz.

(a) Surfaces whose spherical representations are minimal lines

(b) Surfaces whose lines of striction lie on the surface of reference

(c) Principal surfaces

(d) Distributive ruled surfaces

(e) Surfaces $t = \dfrac{2t_1 t_2}{t_1+t_2}$

whose equations are respectively

$$f=0, \phi=0, J(f, \phi)=0, J[f, J(f, \phi)]=0, J[\phi, f(f, \phi)]=0.$$

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ON A CONJECTURE OF MILLER

BY

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1. If $$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r},$$
where the p's are different primes $\geqslant 2$ and each $\alpha \geqslant 1$, then the Mobius function $\mu(n)$ is defined by the relations :

$$\mu(1) = 1,$$
$$\mu(n) = 0 \text{ if any } \alpha \geqslant 2,$$
and $$\mu(n) = (-1)^r \text{ if no } \alpha > 1.$$

We further write

$$M_1(n) = \sum_1^n \mu(m) \text{ and } M_2(n) = \sum_1^n M_1(m).$$

Analogous to the Mobius functions $\mu(n)$ and $M_1(n)$, we have Liouville's functions $\lambda(n)$ and $L(n)$ defined by the relations :

$$\lambda(1) = 1,$$
$$\lambda(n) = (-1)^{\alpha_1+\alpha_2+\alpha_3+\cdot\cdot+\alpha_r},$$
and $$L(n) = \sum_1^n \lambda(m).$$

Polya's conjecture that

$$L(n) \leqslant 0 \text{ for } n \geqslant 2,$$

was verified by me* for values of n up to 20000.

In his letter dated the 11th May, 1948, Dr. J. C. P. Miller stated that

$$A(n) = \{ M_2(n)/n \} \leqslant 0 \text{ for } 3 \leqslant n \leqslant 1000,$$

and asked if $A(n)$ ever became positive. I now find that it does not for values of n up to 20,000 at least. The tables that follow give values of $-M_2(25\ m)$ and $-A(100\ m)$ for values of m up to 800 and 200 respectively. It appears that

$$| M_2(n) | = O(n \log n).$$

* Gupta, H. . On a table of values of $L(n)$, *Proc. Ind. Ac. Sc.*, 12 (1940), 407-9.

2. A Table of Values of $-M_2(25\,m)$:

m	0	1	2	3	4	5	6	7	8	9
		41	93	139	174	251	287	316	431	442
1	451	499	647	719	688	705	731	734	864	1008
2	1132	1218	1207	1083	944	979	1053	1298	1525	1610
3	1673	1740	1760	1725	1778	1793	1791	1777	1692	1618
4	1572	1583	1718	1885	2100	2418	2657	2785	2934	2981
5	3031	3024	3015	3053	3020	2859	2677	2403	2230	2163
6	2129	2159	2164	2240	2367	2638	2977	3251	3516	3788
7	4017	4245	4430	4564	4621	4637	4744	4847	4832	4752
8	4606	4532	4437	4271	4195	4165	4169	4206	4061	3843
9	3612	3429	3384	3408	3405	3489	3688	4023	4319	4557
10	4670	4687	4726	4787	4685	4507	4287	4043	4027	4248
11	4565	4949	5439	6011	6577	7101	7574	7999	8393	8787
12	9016	9177	9307	9394	9454	9313	9184	8947	8768	8485
13	8088	7676	7164	6692	6245	5866	5529	5143	4770	4438
14	4173	3874	3719	3622	3570	3599	3752	3854	4027	4211
15	4447	4609	4773	4949	5125	5372	5638	5864	6224	6582
16	6843	7146	7470	7801	8197	8593	8935	9323	9575	9847
17	10163	10577	10884	11099	11278	11530	11748	11959	12125	12345

18	12458	12595	12699	12801	12840	12854	12876	13022	13106	13078
19	13065	12919	12666	12461	12175	11716	11228	10777	10421	10195
20	10083	10048	10137	10238	10324	10329	10361	10434	10465	10471
21	10495	10468	10467	10533	10481	10304	10019	9787	9544	9333
22	9104	8895	8743	8592	8425	8097	7725	7438	7230	7070
23	6951	6807	6632	6501	6443	6502	6694	6899	6999	7089
24	7146	7108	7031	6990	7024	7092	7192	7329	7383	7563
25	7779	8000	8294	8777	9241	9780	10371	10877	11279	11654
26	12097	12413	12716	13111	13652	14127	14468	14805	15165	15601
27	15998	16388	16814	17172	17541	18003	18485	18944	19344	19815
28	20441	21116	21754	22305	22742	23093	23346	23427	23444	23496
29	23475	23370	23177	23004	22848	22705	22494	22169	21811	21331
30	20894	20556	20287	20161	20214	20278	20239	20154	20194	20314
31	20456	20575	20619	20671	20662	20606	20639	20602	20656	20873
32	21004	20983	20810	20490	20161	19866	19501	18981	18461	17913
33	17523	17100	16697	16296	15839	15297	14824	14244	13717	13228
34	12581	11766	10991	10158	9378	8665	8079	7465	6946	6514
35	6112	5763	5487	5217	5105	5022	4900	4705	4491	4371
36	4312	4292	4233	4196	4141	4045	3931	3909	3849	3873
37	3916	3947	4061	4154	4276	4362	4472	4688	5070	5491
38	6025	6556	7107	7666	8113	8538	8985	9412	9954	10674
39	11514	12304	13115	13979	14975	15977	16983	17919	18833	19700

2. A Table of Values of $-M_2(25\,m)$:—*Contd.*

m	0	1	2	3	4	5	6	7	8	9
40	20409	20979	21432	21855	22223	22608	22998	23474	23969	24496
41	24955	25554	26261	27007	27850	28642	29403	30049	30653	31221
42	31733	32366	33016	33525	33931	34360	34909	35450	35882	36320
43	36838	37331	37795	38185	38431	38661	38906	39094	39305	39539
44	39756	39917	39967	39942	40030	40055	40102	40173	40212	40144
45	40089	40128	40201	40163	40100	39996	39717	39412	39130	38859
46	38555	38283	38055	37760	37364	37018	36653	36128	35533	34947
47	34228	33302	32289	31364	30631	29904	29222	28484	27899	27432
48	27046	26649	26248	25798	25256	24872	24492	24113	23667	23160
49	22642	22185	21701	21161	20719	20312	19986	19653	19283	18908
50	18585	18375	18190	18138	18169	18126	18045	17970	17897	17751
51	17520	17273	16932	16711	16541	16423	16328	16321	16434	16540
52	16721	17029	17339	17635	17904	18191	18411	18614	18916	19119
53	19403	19831	20268	20626	20973	21257	21362	21463	21529	21566
54	21595	21650	21618	21488	21270	21166	21102	20914	20779	20672
55	20705	20699	20737	20790	20772	20714	20834	20964	20953	20771
56	20444	20112	19690	19342	19050	18632	18175	17789	17356	16881
57	16415	15875	15341	14907	14645	14382	13948	13556	13158	12792
58	12376	11959	11622	11552	11478	11330	11240	11234	11175	11056
59	11079	11225	11420	11671	11999	12344	12774	13186	13653	14126

60	14439	14595	14692	14698	14685	14581	14417	14215	13986	13830
61	13728	13656	13771	13937	14204	14417	14647	14803	14984	15207
62	15474	15742	16079	16449	16873	17325	17825	18441	19062	19531
63	20104	20730	21328	22115	22878	23482	24068	24582	25073	25628
64	26293	26961	27591	28265	29052	29882	30850	31828	32750	33641
65	34508	35378	36274	37183	37973	38693	39492	40304	41117	41883
66	42572	43185	43852	44506	45058	45557	46036	46541	47086	47719
67	48219	48756	49170	49595	50166	50587	50914	51318	51806	52254
68	52748	53310	53942	54508	54923	55469	56030	56543	57109	57686
69	58245	58665	59029	59473	59995	60441	60844	61191	61628	62029
70	62489	63075	63571	64038	64454	64826	65187	65353	65462	65483
71	65550	65722	65813	65888	66007	66083	66084	66123	66237	66378
72	66645	66931	67144	67523	67900	68167	68500	68896	69224	69553
73	70061	70679	71172	71663	72193	72752	73300	73847	74361	74852
74	75399	75973	76571	77151	77702	78171	78476	78554	78565	78589
75	78664	78678	78691	78580	78272	77814	77201	76552	75789	74961
76	74037	73055	72002	71013	69997	68796	67690	66648	65492	64409
77	63364	62246	61032	59884	58882	57755	56659	55618	54616	53651
78	52778	52085	51472	50837	50117	49391	48493	47352	46171	45010
79	43827	42690	41558	40468	39325	38188	37200	36205	35258	34341
80	33610	.		..	..	.	..	.	..	..

3. A Table of Values of $-A(100\, m)$:

m	0	1	2	3	4	5	6	7	8	9
		1.7400	2.1550	2.1567	1.8275	2.2640	1.5733	2.1786	2.2000	1.9900
1	1.5720	1.9091	2.4450	2.3192	1.9121	1.4193	1.4794	2.0682	2.4611	2.4968
2	2.3030	1.9976	1.8459	1.4713	1.5367	1.8680	1.8019	1.4915	1.9425	2.6117
3	3.0053	3.0497	2.7400	2.1709	1.6262	1.1923	0.9917	1.0884	1.2561	1.4456
4	1.7108	1.9993	2.2798	2.5312	2.6700	2.7684	2.7913	2.7885	2.6388	2.2914
5	2.0166	2.0243	2.0125	1.9749	1.8554	1.6553	1.5045	1.2684	1.1434	1.1346
6	1.1910	1.1515	1.1908	1.3165	1.6205	1.8611	2.0685	2.2634	2.4726	2.6790
7	2.9201	3.2031	3.2561	3.1749	3.0397	2.7859	2.6597	2.6226	2.6435	2.6125
8	2.6255	2.4890	2.2513	2.0117	1.7648	1.4801	1.0905	0.7984	0.6235	0.5506
9	0.4791	0.4551	0.4184	0.4367	0.4757	0.6342	0.8451	1.0262	1.3383	1.7155
10	2.0409	2.2003	2.3499	2.5496	2.8272	3.0222	3.2010	3.3535	3.4995	3.5694
11	3.6142	3.6063	3.5904	3.5576	3.4840	3.3526	3.2210	3.0370	2.7364	2.4556
12	2.2538	2.0873	1.9399	1.7643	1.6118	1.4868	1.4420	1.4092	1.3228	1.2657
13	1.2862	1.3667	1.4330	1.5239	1.5942	1.5996	1.5640	1.5167	1.5027	1.4988
14	1.4603	1.3511	1.2223	1.0728	0.9686	0.8535	0.7862	0.7602	0.7716	0.8573
15	0.9626	0.9725	0.9201	0.9001	0.9511	0.9983	1.0816	1.2141	1.3499	1.5137
16	1.6433	1.8045	2.0216	2.2254	2.4080	2.5801	2.7143	2.8195	2.9268	3.0127
17	3.1028	3.2119	3.3203	3.4121	3.4968	3.5708	3.6622	3.6984	3.6974	3.6918
18	3.7025	3.7514	3.8035	3.8892	3.9837	4.0756	4.1775	4.2013	4.1857	4.0847
19	3.8967	3.6648	3.4110	3.1623	2.9206	2.7066	2.5570	2.3437	2.0989	1.8693
20	1.6805	..	..	..	..	..	..	..	..	..

SKEW TENSORS AND REGION-COMPLEXES IN S_n*

BY

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1. INTRODUCTION. A skew-symmetric or linear tensor of the $(r+1)$th order $a_{ijk\,\ldots\,l}$ with $(r+1)$ suffixes is the symbol of a multilinear form in $(r+1)$ sets of $(n+1)$-ary variables, skew-symmetric in every pair of them. We will briefly explain how this may be interpreted as a linear R_r-complex, that is, as a linear complex of r-dimensional sub-regions of a projective space S_n of n dimensions. If $(x_0^{(i)}, x_1^{(i)}, \ldots, x_n^{(i)})$ $(i=0, 1, \ldots, r)$ are homogeneous co-ordinates of $(r+1)$ linearly independent points, the determinants $P(i_1, i_2, \ldots, i_{r+1})$† formed from the i_1th, i_2th, .., i_{r+1}th columns of the matrix $\|x_0^{(i)}, x_1^{(i)}, \ldots, x_n^{(i)}\|$ $(i=0, 1, \ldots, r)$, are called the *co-ordinates* of the r-dimensional region $R_r^{(1)}$ determined by the $(r+1)$ points. It is clear that if the $(r+1)$ points are replaced by $(r+1)$ independent linear combinations of them, then these region-co-ordinates $P(i_1, i_2, \ldots, i_{r+1})$ will be only multiplied by a universal common factor.

The r-dimensional region R_r can also be looked upon as the intersection of $(n-r)$ primes (i.e. R_{n-1}'s), $\{w_0^{(i)}, w_1^{(i)}, \ldots, w_n^{(i)}\}$ $(i=1, 2, \ldots, n-r)$. Hence it can also be represented by a system of dual co-ordinates $\bar{P}(i_1, i_2, \ldots, i_{n-r})$, which are defined as the determinants formed by the i_1th, i_2th, .., i_{n-r}th columns of the matrix $\|w_0^{(i)}, w_1^{(i)}, \ldots, w_n^{(i)}\|$ $(i=1, 2, \ldots, n-r)$.

* I wish to thank Dr. R Vaidyanathaswamy for his help and guidance in the preparation of this paper.

† Tensor indices, which are usually written as suffixes, are here written in the same line but within brackets.

It is easy to prove that the region-co-ordinates $P(i_1, i_2, .., i_{r+1})$ are proportional to its dual region-co-ordinates $\bar{P}(j_1, j_2, .., j_{n-r})$, where $(i_1, i_2, .., i_{r+1}, j_1, j_2, .., j_{n-r})$ is an even permutation of $(0, 1, 2, .., n)$.

For, if we choose a set of $(n+1)$ independent points $x^{(i)}$ $(i=0, 1, .., n)$, out of which the first $(r+1)$ points $x^{(j)}$, $(j=0, 1, .., r)$ lie in the region in question and consider the determinant $\Delta = |x_0^{(i)}, x_1^{(i)}, .., x_n^{(i)}|$ $(i=0, 1, .., n)$ and its adjugate $|X_0^{(i)}, X_1^{(i)}, .., X_n^{(i)}|$ $(i=0, 1, 2, .., n)$, then by a well-known theorem(2), the determinant $P(i_1, i_2, .., i_{r+1})$ formed from the first $(r+1)$ rows and the i_1th, i_2th, .., i_{r+1}th columns, multiplied by the factor Δ^{n-r-1}, is equal to the determinant $\bar{P}(j_1, j_2, .., j_{n-r})$, formed from the last $(n-r)$ rows and the j_1th, j_2th, .., j_{n-r}th columns of of the adjugate, where $(i_1, i_2, .., i_{r+1}, j_1, .., j_{n-r})$ is an even permutation of $(0, 1, .., n)$. This proves the proportionality stated.

We may note further that the condition that the R_r with region-co-ordinates $P(i_1, i_2, .., i_{r+1})$ intersects the R_{n-r-1} with the dual region-co-ordinates $\bar{Q}(i_1, i_2, .., i_{r+1})$ is $\Sigma P(i_1, i_2, .., i_{r+1})\ \bar{Q}(i_1, i_2, .., i_{r+1}) = 0$, where the summation is for all indices $i_1, i_2, .., i_{r+1}$ from 0 to n. For, by choosing $(r+1)$ independent points $x^{(i)}$ $(i=0, 1, 2, ..r)$ in R_r and $(n-r)$ independent points in R_{n-r-1}, the condition of intersection is the same as the condition for these points to lie in a prime; that is, it is the vanishing of the determinant formed from the $(n+1)$ points. This condition can be written in the form

$$\Sigma P(i_1, i_2, .., i_{r+1})\ \bar{Q}(i_1, i_2, .., i_{r+1}) = 0.$$

Using $(n+1)$ linearly independent points as a simplex of reference, the points of S_n may be represented in the usual way, by $(n+1)$ homogeneous co-ordinates x^i and dually, the primes by $(n+1)$ prime-co-ordinates w_i, $(i=0, 1, .., n)$, in such wise that by the transformation of co-ordinates, x^i and w_i transform as contravariant and covariant vectors. Since from the point of view of

Projective geometry, constant factors do not make any difference to the geometrical elements, we need not distinguish between tensor-densities and tensors.

It is clear that the region-co-ordinates $P(i_1, i_2, .., i_{r+1})$ of R_r are themselves the components of a skew-symmetric contravariant tensor of order $(r+1)$, which can be obtained as the alternating product of $(r+1)$ contravariant vectors, viz. $r+1$ independent points on the region. Such tensors are called *simple linear tensors.* Similarly the dual region-co-ordinates $\bar{P}(j_1, j_2, .., j_{n-r})$ of R_r are the components of a skew-symmetric covariant simple tensor of order $(n-r)$. These two are dual to each other. For, the dual tensor(3) of $P(i_1, i_2, .., i_{r+1})$ representing the R_r is $\bar{P}(j_1, j_2, .., j_{n-r})$ of the $(n-r)$th order, where $i_1, i_2, .., i_{r+1}, j_1, j_2, .., j_{n-r}$ is an even permutation of $0, 1, 2, .., n$, since

$$\bar{P}(j_1, j_2, .., j_{n-r}) = \frac{1}{(n-r)!} \varepsilon(j_1, j_2, .., j_{n-r}, i_1, .., i_{r+1})$$
$P(i_1, .., i_{r+1})$.

The general covariant linear tensor $a(i_1, i_2, .., i_{r+1})$ of order $(r+1)$ can be geometrically interpreted as the complex Γ_r of all R_r's, whose region-co-ordinates $P(i_1, i_2, .., i_{r+1})$ satisfy the equation $a(i_1, .., i_{r+1})\ P(i_1, .., i_{r+1}) = 0$. If the linear tensor $a(i_1, .., i_{r+1})$ is itself simple, so that the quantities a are the dual region-co-ordinates of a certain R_{n-r-1}, this complex reduces to the complex of all R_r's intersecting the R_{n-r-1}. R_r-complexes of this kind may be called *special* or *axial* complexes with the *axis* R_{n-r-1}. Extreme cases of the axial Γ_r arise for $r = 0$ and $r = n-1$. The former is the set of points on a prime and the latter is the set of all primes though a point.

The purpose of this paper is to obtain certain geometrical interpretations of the skew-product of a linear tensor with a covariant simple tensor and of the contracted product of a linear tensor and a contravariant simple tensor.

If $a_{ij\ldots kl}$ is any tensor, we may, following Schouten[4], denote by $a_{[ij\ldots k]l}$, the part of the tensor which is skew in the suffixes $i, j, \ldots, k$. For example, $a_{i[jk\lambda]\mu} = a_{ijk\lambda\mu} + a_{ik\lambda j\mu} + a_{i\lambda jk\mu} - a_{ikj\lambda\mu} - a_{ij\lambda k\mu} - a_{i\lambda kj\mu}$. This process may also be applied to the products of linear tensors. For example, $a_{i[jk}b_{\lambda]\mu} = a_{ijk}b_{\lambda\mu} - a_{i\lambda k}b_{j\mu} - a_{ij\lambda}b_{k\mu}$. We have here only three terms, since the product already alternates in j and k.

Before we proceed further, we will introduce certain conceptions due to Kantor[5] relating to an R_r-complex*, Γ_r in S_n.

A *singular point* of Γ_r is a point such that all r-dimensional regions through it belong to Γ_r. A *complete prime* of Γ_r is a prime such that every R_r in it belongs to Γ_r.

More generally, an $R_{r-\lambda}$ which is such that all R_r's through it belong to Γ_r is called a "*singular* $R_{r-\lambda}$" of Γ_r. If all R_r's lying in a certain $R_{r+\lambda}$ belong to Γ_r, the $R_{r+\lambda}$ is called a "*complete* $R_{r+\lambda}$" of Γ_r.

We immediately get

THEOREM I. *The join of two singular points of* Γ_r *is composed entirely of singular points of* Γ_r *; dually, any prime through the intersection of two complete primes of* Γ_r *is also a complete prime of* Γ_r.

For, suppose that P, Q are two singular points of Γ_r and that O is any point on PQ. Take any R_r through O. In this R_r, choose an R_{r-1} not passing through O. Join this R_{r-1} to P and to Q. Since P and Q are singular

* In addition to Kantor's paper in *Crelle's journal* listed in the Bibliography (which though dealing with line-complexes has a lengthy introduction dealing with general linear R_r-complexes), I have seen a subsequent posthumous paper of his on "Systeme linearer Strahlenkomplexe im R_r" Sitzungsberichten der kaiserl, *Akademie der Wissenschaften in Wien*, Bd cxii Juli 1903. I have not been able to get at any later paper of Kantor bearing on the general linear R_r-complex. Probably there are none. Nor, am I aware of any other writer, who has dealt with the general linear R_r-complex.

points, these two joins belong to Γ_r and the R_r through O being a linear combination of these, belongs to Γ_r. Hence O is a singular point.

To prove the second part, let p, q be two complete primes of Γ_r and π any prime through their intersection R_{n-2}. Any R_r in π must intersect this R_{n-2} in an R_{r-1}. Let O be a point of the R_r not in this R_{r-1} and let any line through O cut p, q in P and Q. Then the given R_r is a linear combination of the joins (R_{r-1}, P), (R_{r-1}, Q) both of which belong to Γ_r, since p, q are complete primes. Hence the R_r in π belongs to Γ_r, so that π is a complete prime.

COR. *Hence the totality of singular points of a* Γ_r *constitutes a flat space. Similarly, the totality of complete primes of* Γ_r *consists of all primes through a certain region.*

If r of the $(r+1)$ points determining an R_r are fixed, it is clear that the equation of the R_r-complex Γ_r becomes linear in the co-ordinates of the $(r+1)$th point. Hence complex-R_r's, which contain a given R_{r-1}, lie in a definite prime through R_{r-1}, which is called the *null prime* of the R_{r-1}.

Similarly, given an R_{r+1}, the complex-R_r's contained in it, constitute a linear R_r-complex in R_{r+1}. But a linear complex of primes in any space must consist of all primes through a particular point. Hence the R_r's in R_{r+1}, which belong to Γ_r, pass through a definite point in R_{r+1}, which is called the *null point* of R_{r+1} with respect to the complex Γ_r.

As an example of the corollary of Theorem 1, if an R_r-complex is axial, then its axis-R_{n-r-1} is composed of singular points and every prime through the R_{n-r-1} is a complete prime.

Conversely we have

THEOREM 2. *If a* Γ_r *has an* R_{n-r-1} *composed of singular points, it is axial with the* R_{n-r-1} *as its axis ; dually, if every*

prime through a certain R_{n-r-1} is a complete prime of Γ_r, then Γ_r is axial and has the axis R_{n-r-1}.

For, every R_r intersecting R_{n-r-1} belongs by hypothesis to Γ_r. On the other hand, all such R_r's constitute a linear R_r—complex. Hence these two complexes must be identical. To prove the second part, every R_r in a prime w passing through R_{n-r-1}, belongs to Γ_r. Every such R_r intersects this R_{n-r-1}; conversely every R_r intersecting this R_{n-r-1} lies in a prime w, through the latter. Hence Γ_r must be an axial complex, with R_{n-r-1} as its axis.

We can however prove a stronger result, namely,

THEOREM 3. *If an R_r—complex has every point on a certain R_{n-r-2} as a singular point, then the Γ_r is axial and its axis-R_{n-r-1} contains this R_{n-r-2}. Dually, if Γ_r has every prime through a certain R_{n-r} as a complete prime, then it is axial and its axis-R_{n-r-1} is contained in this R_{n-r}.*

For, choose a simplex (0, 1, . ., n) of which the last ($n-r-1$) points determine the R_{n-r-2}. Let the null point of the R_{r+1} (0, 1, . ., $r+1$) w.r t. Γ_r be the point P. Without loss of generality, P may be taken as the vertex 0 of the simplex. We shall now prove that 0 is also a singular point of the Γ_r. To prove this, it is sufficient to show that all r-dimensional faces of the simplex (0, 1, . ., n) through the vertex 0, belong to Γ_r. Since 0 is the null point of (0, 1, . ., $r+1$), it is clear that all r-dimensional faces with one vertex 0 and the remaining vertices chosen from the points (1, . . , $r+1$) belong to Γ_r. The remaining r-dimensional faces which contain 0 must contain, one atleast of the ($n-r-1$) points ($r+2$, . . , n). Since these points are singular, these remaining r-dimensional faces must necessarily belong to Γ_r. Hence 0 is a singular point. Therefore by Theorem 1, the R_{n-r-1}, which joins 0 and R_{n-r-2}, is composed of singular points; whence by Theorem 2, Γ_r must be axial.

To prove the second part, choose a simplex (0, 1, . ., n) of which the last ($n-r+1$) points determine

the R_{n-r}. Let the null prime of the R_{r-1} $(0, 1, \ldots, r-1)$ be the prime π, intersecting R_{n-r} in a certain R_{n-r-1}. We may suppose that of the $(n-r+1)$ points of the simplex in R_{n-r}, $(n-r)$ points $(r, \ldots, n-1)$ have been chosen in this R_{n-r-1}, while the point n is in the R_{n-r} outside the R_{n-r-1}. We shall prove that π is a complete prime of Γ_r. To prove this, it is sufficient to prove that the nc_{r+1} r-dimensional faces of the simplex $(0, 1, \ldots, n-1)$ all belong to Γ_r. Since π is the null prime of $(0, \ldots, r-1)$, the $(n-r)$ faces $(0, 1, \ldots, r-1, i)$ $(i=r, r+1, \ldots, n-1)$ belong to Γ_r. Any other r-dimensional face of the simplex $(0, 1, \ldots, n-1)$ will leave out one of the points $0, 1, \ldots, r-1$ and therefore will be contained in a prime through R_{n-r}. Since by hypothesis, every such prime is a complete prime of Γ_r, it follows that every r-dimensional face of the simplex $(0, 1, \ldots, n-1)$ belongs to Γ_r, so that π is also a complete prime of Γ_r. Any prime through the R_{n-r-1} can be exhibited as a linear combination of π and a prime through R_{n-r}. Hence by Theorem 1, every prime through R_{n-r-1} is a complete prime. Hence by Theorem 2, the complex Γ_r is axial with the axis R_{n-r-1}, contained in the R_{n-r}.

3. As explained before, we think of a covariant linear tensor $a_{ijk..l}$, with $(r+1)$ indices $i, j, k, \ldots, l$ as representing a linear R_r-complex Γ_r in S_n. The contracted product of this tensor with a point x^i, is a tensor of the rth order, $a_{ijk \ldots l}\, x^i$ and therefore represents a certain R_{r-1}-complex Γ_{r-1}, which we call the *x-contract* of Γ_r. Similarly, the alternating product of the linear tensor with a prime w is the linear tensor $a_{[ijk \ldots l}w_{\lambda]}$ of order $(r+2)$ and represents a certain R_{r+1}-complex Γ_{r+1}, which we call the *w-extension* of Γ_r.

THEOREM 4. *The x-contract* Γ_{r-1}, *of the complex* Γ_r, *is the totality of all* R_{r-1}*'s, whose join with x belongs to the given complex* Γ_r. *The w-extension* Γ_{r+1} *of* Γ_r, *represents the complex of all* R_{r+1}*'s, which intersect the prime w in an* R_r *of the given complex* Γ_r.

To prove this, we can take x as the vertex $(1, 0, \ldots, 0)$ of the simplex of reference and w as the prime face of the simplex opposite to this vertex. Hence the x-contract Γ_{r-1} of Γ_r is given by $b_{jk.\,l} = a_{ijk\;\;l}\, x^i = a_{0jk.\,l}$.

If an R_{r-1} has the region-co-ordinates $P_{jk\;\;l}$, then the region-co-ordinates $\pi_{ijk\;\;l}$ of the R_r, which is the join of R_{r-1} to $(1, 0, \ldots, 0)$ are given by:

$\pi_{0jk..l} = P_{jk\,.l}$; $\pi_{ijk\;\;l} = 0$, if none of the suffixes is 0. Hence the R_{r-1}'s whose join with $(1, 0, \ldots, 0)$ belongs to Γ_r are determined by $a_{ijk.\,l}\, \pi_{ijk\;\;l} = 0$,

or $$a_{0jk..l}\, P_{jk\;\;l} = 0,$$
which may also be written $b_{jk\;\;l}\, P_{jk\;\;l} = 0$. Thus the R_{r-1}'s with this property are precisely those, which belong to the x-contract of Γ_r.

For the second part of the theorem, let $b_{\lambda ijk.\,l}$ represent the w-extension of $a_{ijk\;\;l}$, so that

$$b_{\lambda ijk.\,l} = a_{[ijk\;\;l}\, w_{\lambda]}.$$
Since we have taken w_λ to be $(1, 0, \ldots, 0)$, we have $b_{0ijk\;\;l} = a_{ijk\;\;l}$ when $i, j, k, \ldots, l$ are all different from zero; $b_{\lambda ijk.\,l} = 0$, if none of the suffixes is 0. Hence the w-extension is the complex of all R_{r+1}'s, whose co-ordinates $\pi_{\lambda ijk\;\;l}$ satisfy $b_{\lambda ijk\;\;l}\, \pi_{\lambda ijk\;\;l} = 0$. Using the fact, that $b_{\lambda ijk..l}$ vanishes when all its suffixes are different from zero, this equation can be written in the form $b_{0ijk\;\;l}\, \pi_{0ijk\;\;l} = 0$. Now it is easily seen that the co-ordinates of the R_r $P_{ijk\,.l}$, which is the intersection of the R_{r+1}, $\pi_{\lambda ijk.\,l}$ with

$w_\lambda(1, 0, \ldots, 0)$ are given by $P_{ijk\ldots l} = \pi_{0ijk\ldots l}$. Since $b_{0ijk\ldots l} = a_{ijk\ldots l}$, the equation of the w-extension simply expresses that the intersection of the R_{r+1}, $\pi_{\lambda ijk\ldots l}$, with w_λ belongs to Γ_r.

COR. 1. *The point x is always a singular point of the x-contract Γ_{r-1} of Γ_r and the prime w is always a complete prime of the w-extension Γ_{r+1} of Γ_r.*

This is obvious from the definition; for any R_{r-1} through x lies in a complex-R_r through x and therefore belongs to Γ_{r-1}; thus x is a singular point of the x-contract. Similarly any R_{r+1} in w contains a complex-R_r and therefore belongs to Γ_{r+1}; thus w is a complete prime of the w-extension.

COR. 2. *If x is a singu'ar point of Γ_r, then the x-contract of Γ_r vanishes identically and conversely. If w is a complete prime of Γ_r, then the w-extension of Γ_r vanishes identically and conversely.*

For, if the x-contract of Γ_r vanishes, any R_{r-1} joined to x belongs to Γ_r, so that x is a singular point of Γ_r. The converse also is obvious. The dual proof holds for the second case.

COR. 3. *If a certain Γ_{r-1} has a singular point x, then Γ_{r-1} may be exhibited as the x-contract of a certain Γ_r. Dually, if w is a complete prime of an R_{r+1}-complex Γ_{r+1}, then the Γ_{r+1} may be exhibited as the w-extension of a certain Γ_r.*

For, if a Γ_{r-1} has a singular point $x^i = (1, 0, \ldots, 0)$, the corresponding linear tensor $b_{jk\ldots l}$ with r indices must vanish, if any one of its indices $= 0$. Hence $b_{jk\ldots l}$ may be considered as the x^i-contract of any complex $a_{ijk\ldots l}$, with $a_{0jk\ldots l} = b_{jk\ldots l}$. The R_r-complex $a_{ijk\ldots l}$ is thus not unique. For the second part, take w as the prime

$(1, 0, .., 0)$ and $b_{\lambda ijk..l}$ with $(r+2)$ suffixes as the Γ_{r+1}. The condition that w is a complete prime of Γ_{r+1} gives $b_{\lambda ijk..l} = 0$, if none of the suffixes equals zero. If we define a tensor $a_{ijk..l}$ subject to the condition that when $i, j, k, .., l$ are different from zero, $a_{ijk..l} = b_{oijk..l}$, then the given Γ_{r+1} is the w-extension of any Γ_r, corresponding to such a tensor $a_{ijk..l}$.

COR. 4. *A singular point y of Γ_r continues to be a singular point of its x-contract $(x \neq y)$. A complete prime u of Γ_r continues to be a complete prime of its w-extension $(u \neq w)$.*

For, since y is a singular point of Γ_r, any R_r through y belongs to Γ_r. Hence any R_{r-1} through y joined to x, is an R_r through y and hence belongs to Γ_r. Hence by the definition of x-contract, any R_{r-1} through y belongs to the x-contract of Γ_r or y is a singular point of the x-contract.

To prove the second part, since u is a complete prime of Γ_r, any R_r in u belongs to Γ_r. Hence any R_{r+1} in u, intersects w, in an R_r in u, which therefore belongs to Γ_r. Hence by the definition of w-extension, any R_{r+1} in u belongs to the w-extension of Γ_r or u is a complete prime of the w-extension.

COR. 5. *The contracts as well as extensions of an axial complex are all axial.*

For an axial Γ_r consists of all R_r's intersecting the axis-R_{n-r-1}; this R_{n-r-1} is composed of singular points of Γ_r. Hence by Cor. 4, all these points together with x are singular points of the x-contract Γ_{r-1} of Γ_r. By Theorem 1, the R_{n-r} got by joining R_{n-r-1} to x consists entirely of singular points of Γ_{r-1} By Theorem 3, it follows that Γ_{r-1} is the axial complex, with this R_{n-r} as its axis.

To prove the other part, an axial Γ_r consists of all R_r's intersecting the axis-R_{n-r-1}; this R_{n-r-1} is such that every prime through it is a complete prime. Hence by

Cor. 4, these primes together with w are complete primes for the w-extension Γ_{r+1} of Γ_r. By Theorem 1, every prime through the R_{n-r-2} got by the intersection of R_{n-r-1} with w is a complete prime. By Theorem 3, it follows that Γ_{r+1} is an axial complex with this R_{n-r-2} as its axis.

COR. 6. *The repeated contracts of Γ_r with r points gives the null prime of the R_{r-1}, which contains these r points. Dually, the repeated extension of Γ_r with $(n-r-1)$ primes gives the null point of the R_{r+1}, which is the intersection of these $(n-r-1)$ primes.*

For, from Theorem 4, it easily follows that the repeated contracts of Γ_r with s independent points defining an R_{s-1}, gives the linear R_{r-s}–complex of all R_{r-s}'s, whose join with these s points is an R_r belonging to Γ_r. Hence if $s=r$, the repeated contracts is the complex of all points, whose join with R_{r-1} is an R_r of Γ_r. This is by definition the null prime of R_{r-1}.

Similarly from the interpretation of the w-extension, it easily follows that the repeated extensions of Γ_r with s independent primes defining a R_{n-s} gives the linear complex Γ_{r+s} of all R_{r+s}'s, whose intersection with R_{n-s} is an R_r of Γ_r. Hence if $s=n-r-1$, the repeated extension is the complex of primes cutting R_{r+1} in an R_r of Γ_r. These are the primes passing through the null point of R_{r+1}.

4. We now concern ourselves with finding the geometrical interpretations of the vanishing of the skew-product (contracted product) of a linear covariant tensor $a(i_1, i_2, \ldots, i_{r+1})$ of the $(r+1)$th order with a simple covariant tensor $\bar{P}(j_1, j_2, \ldots, j_s)$ representing a R_{n-s}, (a simple contravariant tensor $P(i_1, j_2, \ldots, j_s)$).

THEOREM 5. $a([i_1, i_2, .., i_{r+1})\ \bar{P}(j_1], j_2, .., j_s) \equiv 0$ *means that every prime through* $\bar{P}(j_1, j_2, .., j_s)$ *is a complete prime of* $a(i_1, i_2, .., i_{r+1})$.

Dually, $a(i_1, i_2, .., i_{r+1})\ P(i_1, j_2, .., j_s) = 0$ *means that every point on* $P(i_1, j_2, .., j_s)$ *is a singular point of the complex* $a(i_1, i_2, .., i_{r+1})$.

For, $a([i_1, i_2, .., i_{r+1})\ \bar{P}(j_1], j_2, .., j_s) = 0$ is equivalent to the statement that $a([i_1, .., i_{r+1})\ \bar{P}(j_1], j_2, .., j_s)\ \pi(j_2, .., j_s) = 0$, for every contravariant tensor π of order $(s-1)$ or in particular for every simple tensor π (or R_{s-2}).

For, let $\bar{P}(j_1, j_2, .., j_s)\pi(j_2, .., j_s) = w(j_1)$, so that $a([i_1, i_2, .., i_{r+1})\ w(j_1]) = 0$. Now $\bar{P}(j_1, j_2, .., j_s)$ as a simple covariant linear tensor, represents an axial R_{s-1}—complex whose axis is the R_{n-s} with the dual co-ordinates $\bar{P}(j_1, j_2, .., j_s)$. By Cor. 6, Theorem 4, $w(j_1)$ is the prime, which joins $\bar{P}$ and π. As the region π is arbitrary, $w(j_1)$ is an arbitrary prime through $\bar{P}$. Hence by Theorem 4, it follows that every prime through $\bar{P}$ is a complete prime of the complex $a\ (i_1, .. i_{r+1})$.

To prove the second part, we note that $a(i_1, .., i_{r+1})\ P(i_1, j_2, .., j_s) = 0$ is equivalent to saying that

$$a(i_1, i_2, .., i_{r+1})\ P(i_1, j_2, .., j_s)\ \bar{\pi}(j_2, .., j_s) = 0,$$

or every simple covariant tensor $\bar{\pi}$ of the $(s-1)$th order. By a dual argument $P(i_1, j_2, .., j_s)\ \bar{\pi}(j_2, .., j_s)$ represents a point x'^1, which is the intersection of P and $\bar{\pi}$. Hence the contracted product reduces to $a(i_1, .., i_{r+1})\ x'^1 = 0$, where x'^1 is an arbitrary point on P, since $\bar{\pi}$ is arbitrary. By Theorem 4, it follows that every point on P is a singular point of $a(i_1, .., i_{r+1})$.

An important case of the above theorem is given by

THEOREM 6. *If* $a([i, .., i_{r+1})\ \bar{P}(j_1], j_2, .., j_r) \equiv 0$, *then* $a(i_1, .., i_{r+1})$ *is axial and its axis-*R_{n-r-1} *is contained in*

$\bar{P}(j_1, j_2, .., j_r)$. *Dually, if* $a(i_1, i_2, .., i_{r+1})$ $P(i_1, j_2, .., j_{n-r-1}) = 0$ *then* $a(i_1, .., i_{r+1})$ *is axial and its axis-*R_{n-r-1} *contains* $P(i_1, j_2, .., j_{n-r-1})$.

For, by the previous theorem, every prime through the R_{n-r} $\bar{P}$ is a complete prime of $a(i_1, .., i_{r+1})$. Hence by Theorem 3, the Γ_r is an axial complex and its axis-R_{n-r-1} is contained in $\bar{P}$.

To prove the second part, by the previous theorem, every point on the R_{n-r-2} P is a singular point of Γ_r. Hence by Theorem 3, the Γ_r is an axial complex and its axis-R_{n-r-1} contains P.

5. The last two theorems were concerned with the geometrical interpretations of the identical vanishing of the skew or contracted product of a covariant linear tensor of order $(r+1)$ with a simple covariant or contravariant tensor of order s, in which only one suffix of the latter entered into the alternation or contraction. More generally, we may enquire what are the geometrical interpretations of the identical vanishing of the alternated or contracted products, when k indices enter into the alternation or contraction. For $k=s$, the answer is contained in Cor. 6, Theorem 4 and its proof; namely, $a([i_1, .., i_{r+1})$ $\bar{P}(j_1, .., j_s]) \equiv 0$, if and only if, the R_{n-s} defined by $\bar{P}$ is a complete region of Γ_r, and $a(i_1, .., i_{r+1})$ $P(i_1, .., i_s) = 0$, if and only if, the R_{s-1} defined by P is a singular region of Γ_r.

For the partial extensions and contracts, we adopt the same device as in the proof of Theorem 5. It will follow in the case of alternating k of the s indices, that $a([i_1, .., i_{r+1})$ $\bar{P}(j_1, .., j_k], j_{k+1}, .., j_s) \equiv 0$ is equivalent to saying that

$a([i_1, \ldots, i_{r+1})\bar{P}(j_1, \ldots, j_k], j_{k+1}, \ldots, j_s)\ \pi(j_{k+1}, \ldots, j_s) \equiv 0,$ for every region π of $(s-k-1)$ dimensions. This is equivalent to the statement that

$a([i_1, \ldots, i_{r+1})\ \bar{\pi}(j_1, \ldots, j_k) \equiv 0,$ for every region $\bar{\pi}$ of $(n-k)$ dimensions passing through the region $\bar{P}$ of $(n-s)$-dimensions. In other words, it is equivalent to the statement that every R_{n-k} through $\bar{P}$ is a complete region of the given complex. Similarly in the case of contracting k of the indices, we see that the statement:

$$a(i_1, \ldots, i_{r+1})\ P(i_1, \ldots, i_k, j_{k+1}, \ldots, j_s) \equiv 0$$

is equivalent to the statement that $a(i_1, \ldots, i_{r+1})$ $\pi(i_1, \ldots, i_k) \equiv 0$, for region π of $(k-1)$-dimensions contained in the region P of $(s-1)$ dimensions. In other words, this is equivalent to saying that every R_{k-1} contained in P is a singulur region of the complex.

It is seen from the geometrical interpretations, that the cases where k has different values are not mutually equivalent. The vanishing of the alternated and contracted products, in which only one suffix is alternated or contracted implies the vanishing of the products, when more than one suffix is alternated or contracted but not vice-versa.

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